

Ambiguity in Corporate Finance*

Preliminary and Incomplete

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Ambiguity in Corporate Finance

Abstract

We study the effect of ambiguity on corporate financial decisions using a non-Bayesian multiprior approach. We consider two models of decision making in the presence of ambiguity, Minimum Expected Utility (MEU) due to Gilboa and Schmeidler (1989) and Consensus Expected Utility (CEU) due to Bewley (2002). With these perspectives we reconsider a canonical corporate finance model in a world of self sufficiency and also in a world with outside financing. In the self sufficiency setting the self-sufficient ambiguity averse corporation may be cautious in exercising expansion options but reluctant to abandon pre-existing assets. When external financing is needed we consider cases where corporations see either more or less ambiguity than financiers. The contract solution to ambiguity is very different from the solution to asymmetric information even though they both facilitate exchange when agents have different beliefs. We also add to the decision theory literature by suggesting how corporate decisions can be used to infer which of various approaches to ambiguity aversion is more empirically relevant. The analysis highlights the importance of empirically identifying the status quo and the process by which consensus is reached in studying investment behavior.

1 Introduction

Most of standard finance theory is built on the Bayesian paradigm of subjective expected utility (SEU) axiomatized by Savage (1954). Despite being the dominant paradigm for the study of decision making, however, it is well-known that the SEU paradigm is not equipped to deal with several phenomena observed in experimental studies. Seminal among these studies are the thought experiments of Ellsberg (1961) that highlight how ambiguity¹ affects individuals' willingness to bet.² Although Ellsberg's experiments and several other experimental papers emphasize the subjects' *aversion* to ambiguity, there are also studies that have shown situations in which subjects are ambiguity *loving*, as in Heath and Tversky (1991) "competence hypothesis" (see Luce (2000) for an extensive survey of the experimental evidence). Regardless of the direction of the impact, it is clear that the existing evidence in several fields of study suggests that ambiguity and attitudes toward ambiguity are important for choice.

In this paper we study how ambiguity affects corporate finance. Since corporations allocate resources across ambiguous and unambiguous investment opportunities, understanding how ambiguity affects corporate decisions is essential to understanding allocative efficiency. Moreover, since real decisions directly influence the dynamics of returns and risk, this understanding is potentially important to the understanding of asset prices.

We follow a literature that captures ambiguity through a non-Bayesian multiprior approach to decision making. There are a number of specific approaches to decision making with multiple priors that have been developed and from these we select two prominent ones: Maximin Expected Utility (MEU) due to Gilboa and Schmedler (1989) and Consensus Expected Utility (CEU) due to Bewely (2002). From these perspectives we reconsider a canonical corporate finance model of real investment and financing decisions in a world of autarky and a world with exchange.

The canonical corporate finance model at the heart of our analysis has a very simple structure. An entrepreneur (E) has a potentially economically valuable idea but lacks the required

¹Ellsberg refers to ambiguity as "a quality depending on the amount, type, reliability and 'unanimity' of information, and giving rise to one's 'degree of confidence' in an estimate of relative likelihoods." (Ellsberg, 1961, p. 657).

²One of Ellsberg's experiments involves two urns with 100 balls each. In the first urn, the unambiguous urn, the subject is told that there are 50 white and 50 blue balls. In the second urn, the ambiguous urn, no information is given on the proportion of white and blue balls. The subject has to choose an urn and a color. After that, a ball will be drawn from the chosen urn and a prize will be awarded if the drawn ball is of the chosen color. The vast majority of subjects chooses to place either of the bets (blue or white ball) on the unambiguous urn. This behavior cannot be justified by any probability distribution since it implies that the subject believes that the probability of a specific outcome (drawing a blue ball from the ambiguous urn) is both less than and greater than 50%.

investment resources while a financier (F) has the required resources but does not have direct access to investment opportunities. This problem has been extensively studied in the Bayesian paradigm of subjective expected utility (SEU) axiomatized by Savage (1954). The fundamental implication of the SEU paradigm is that individuals select among actions with risky outcomes by attaching a utility index to each outcome and a *unique* probability to the likelihood that the outcome will obtain. The decision maker then choose actions that maximize expected utility.

Finance researchers have explored alternative models of rationality or 'behavioral models' although the departure from SEU is not always clear. For the most part, the SEU framework is maintained but it is assumed that agents do not follow Bayes rule in incorporating evidence or experience into their decisions.³ We also explore non-Bayesian behavior but do so through an explicit departure from SEU. Specifically, we follow decision theorists who build their analysis on the relaxation of specific savage axioms. We consider two such cases both of which involve representations of preferences in which a *set* of probabilities (instead of a unique probability) is involved.⁴

Perhaps the most widely cited alternative to SEU is the "Maxmin expected utility with multiple priors" (MEU) of Gilboa and Schmeidler (1989), in which the Savage Axiom of 'Independence' is relaxed. The result is that decisions are characterized by the prior that delivers the lowest expected utility. A very attractive feature of MEU preferences is that they can "explain" the decisions observed in the Ellsberg's experiments.

An alternative decision rule based on multiple priors is the "Consensus Expected Utility" (CEU) proposed by Bewley (2002) as a formalization of Knight's idea of uncertainty. This preference representation can be obtained from SEU by removing the assumption that preferences are complete (i.e., any two gambles can always be ranked) and replace it with an assumption of *inertia* (i.e., a gamble is never accepted unless acceptance is preferred to rejection under all prior distributions). Bewley (2002) shows that by removing the assumption of completeness from SEU one obtains a *set* of probability distributions that allows characterization of the choice of an agent according to a "unanimity rule": a gamble is preferred to another if and only if its

³see Baker and Wurgler for a survey of Behavior Corporate Finance.

⁴An alternative representation characterizes agents' beliefs through expected utility in which expectations are computed with respect to a non-additive probability (capacity), as in the "Choquet Expected Utility" of Schmeidler (1989).

expected value is higher under *all* possible probability distributions.⁵ Because these unanimity preferences are incomplete, there does not exist a numerical index that represents them, making them unsuitable for optimization problems on which large part of economic decision making is based. To complete the model, Bewley (2002) imposes the assumption of inertia under which a person remains with the status quo unless an alternative is deemed unanimously better.⁶ It is important to note that in Bewley’s characterization, the status quo is not considered a behavioral *bias* but a device to complete the preferences. The treatment of the status quo is conceptually different in the behavioral economic literature that relies on “reference-dependent” preferences to analyze the implications of biases such as the endowment effect, loss aversion or framing.⁷

We feel an examination of CEU is especially appropriate in the corporate context due to its correspondence to a unanimity rule. That is, CEU can be thought of as a model of a group of decision makers, perhaps a board of directors or a management team, where each member of the group has a set of priors. If this group only accepts proposed actions that are acceptable to all, they will act as if they were a single agent CEU decision maker.

Several other approaches to decision making in the presence of ambiguity have been examined in more general settings, as Gilboa and Marinacci (2011) discuss in their excellent survey. It is, however, beyond the scope of our paper to explore all of these approaches as we are primarily concerned with basic investment and financing issues in the presence of ambiguity. As a result, we examine the entrepreneur’s problem in the context of SEU, MEU and CEU only.

In the autarkic setting we find that the self-sufficient ambiguity averse entrepreneur may be cautious in exercising expansion options but reluctant to abandon pre-existing assets, even when abandonment payoffs are relatively large. In an exchange setting where the entrepreneur must raise external financing we examine cases where entrepreneurs alternately see more and less ambiguity than financiers. Here we find that, although both agents have differing priors, the

⁵Incomplete preferences were studied originally by Aumann (1962). Dubra, Maccheroni, and Ok (2004) show that an incomplete order over (unambiguous) lotteries can be represented via a multi-utility representation. Kraus and Sagi (2006) extend their analysis to the case of temporal lotteries and provide a model of inter-temporal choice with incomplete preference that generalize Kreps and Porteus (1978) recursive utility. Ghirardato, Maccheroni, and Marinacci (2004) and Gilboa, Maccheroni, Marinacci, and Schmeidler (2010) provide a general form of the Bewley’s representation theorem. Ortoleva (2010) provides a different perspective on Bewley’s inertia assumption and shows how by starting with a *complete* set of preferences that exhibit *status quo bias* one can obtain incomplete preferences as in Bewley.

⁶Gilboa, Maccheroni, Marinacci, and Schmeidler (2010) suggest another approach to obtain complete preferences that relies on imposing axioms for both the unanimous and the MEU preferences and show that MEU can be seen as a completion of unanimous incomplete preferences.

⁷See, for example, Kahneman and Tversky (1979), Kahneman, Knetsch, and Thaler (1991), and Tversky and Kahneman (1991). Sagi (2006) studies the implications of imposing a “no-regret” axiom on a family of complete preferences “anchored” at the status quo.

contract design is considerably different from the contract that would obtain under asymmetric information. We also add to the decision theory literature by pointing out how corporate decisions can be used to infer which of various approaches to ambiguity aversion is more strongly supported by the data.

Our analysis delivers a number of novel interesting findings. In the context of real investment decisions, we show that MEU and CEU are observationally equivalent when an entrepreneur faces the decision to expand an existing venture but have opposite predictions when he is faced with the decision to shut down operations. Specifically, while MEU implies a “pessimistic” decision rule in both expansion and shut-down decisions, CEU entrepreneurs are pessimistic in expansions but *optimistic* in contractions. Because of the unanimity nature of CEU preferences, entrepreneurs expand only if the scaled up venture is better under the worst possible scenario (similar to MEU). However, in contractions, for a CEU entrepreneur the worst case scenario is getting rid of a venture that is very profitable, i.e., CEU takes into account the opportunity cost or potential regret of an action. This implies that, contrary to MEU entrepreneurs, an CEU entrepreneur may continue operating a project even when they believe the project may be worth less than the scrap value of the asset. This finding is in sharp contrast with Miao and Wang (2011) who apply MEU to study expansion and contraction decisions. In essence, ambiguity has a symmetric effect on expansion and contraction options in the MEU setting while it has asymmetric effect in an CEU setting: MEU entrepreneurs are quick to expand and quick to contract, CEU entrepreneurs are quick to expand and *slow* to contract.

A direct implication of the previous finding is that the effect of ambiguity aversion in a MEU real option model cannot be distinguished by a volatility or risk aversion effect in an equivalent SEU setting. A lower volatility (higher risk aversion) in this case reduces both the option to expand and contract, which, therefore will be exercised earlier. On the other hand, the asymmetric effect we observed in the CEU setting cannot be obtained by a change in volatility (or risk aversion) in a SEU setting. This is a potentially important channel that can help to empirically identify which approach best describes investment decisions.

When studying the optimal contract in a static financing setting between a risk-neutral, multi-prior financier and a risk neutral, single-prior entrepreneur, we find that MEU and CEU are observationally equivalent. Both models predict a preference for “debt financing”. Intuitively, E feels that only a single prior is appropriate but knows that F needs to be compensated for

the multiple priors. The amount of compensation paid to F is minimized when F is offered a contract with common payoff in all possible states and hence “immune” to the ambiguity that F sees. The intuition for this finding is however similar to what one would find in the case of SEU with a risk averse F and so, observationally, MEU, CEU and SEU with a risk averse F do not seem to be distinguishable. Interestingly though, the optimal contract under SEU when both E and F are risk neutral but have different beliefs (e.g., E optimist and F pessimist) *never* involves debt as an optimal contract. In this case the optimist E wants to offer a contract that pays a lot in the state he feels least likely to occur.

When studying the optimal contract in a static financing setting between a risk-neutral, multi-prior entrepreneur and a risk-neutral, single-prior financier, we find that the set of feasible contract under MEU is always a strict subset of those feasible under CEU. In particular, because of E 's ambiguity aversion under both MEU and CEU, equity is always an optimal contract. However, while equity is the *unique* optimal contract for MEU, several other non-equity-like contract will be accepted as a financing arrangement by an ambiguity averse entrepreneur with CEU preferences.

Finally, we make a first attempt at studying the problem of financing in a dynamic setting. Because of the complexity of the issues that emerge when dealing with ambiguity in a dynamic setting, we limit our analysis to the special cases of a multi-prior financier who is offered option-like instruments and study his optimal exercise decisions under both MEU and CEU. Specifically, we suppose F is offered a choice between two hybrid securities A and B , where A (B) offers ownership of a security X (Y) with the option to convert to security Y (X) a period later. We find that, while under SEU and MEU the proceeds from issuing A and B are identical, they can be different under CEU, due to the inertia effect on the exercise choice when X and Y are not comparable. This suggests that under CEU, contract design has to carefully consider the *sequencing* of payoffs that a security offers. Furthermore, we find that when F can commit to a particular exercise policy, the value of a security can be different from its value in the absence of commitment. This also point to an important issue about dynamic consistency that needs to be further investigated in the solution of an optimal contract in a dynamic setting.

In summary, our findings indicate that ambiguity and attitudes towards ambiguity can have significantly different implication on real and financial decision, depending on the model a researcher uses to accommodate these deviations from the Bayesian paradigm. These results

therefore suggest caution when relying on ambiguity aversion to “explain” empirically observed phenomena.

Recent studies have applied the multiple prior approach to finance problems. For the most part, these applications have been in the asset pricing and portfolio choice areas. Epstein and Schneider (2010) provide an excellent survey. Fewer studies however have considered how ambiguity aversion could affect corporate decisions. For example, Miao and Wang (2011), Nishimura and Ozaki (2007), Riis Flor and Hesel (2011), and Riedel (2009) examine the exercise decision in a real option setting but do so only in an MEU framework and do not consider financing issues. Our contribution is to study the effect of different modelling choices for ambiguity aversion (MEU vs. CEU) on real and financial decisions.

Rigotti (2004) is, to the best of our knowledge, the first paper that addresses financing issues in the context of the incomplete CEU type preferences.⁸ We complete and extend his analysis by considering both static and dynamic settings as well as real decisions. Moreover, by drawing out investment and financing implications under both MEU and CEU we open the possibility of using actual investment and financing decisions to address the empirical question of which approach best describes managerial decision making.

In the next section we set out some preliminary elements of our analysis and follow this with an examination of a two period real option decision. We characterize the solution of the investment problem and show that different representations of ambiguity lead to stark differences in observed investment behavior. In section 3 we consider the time inconsistency in real decisions that can be present in the face of ambiguity. In particular we show that a decision maker might be willing to invest if they could precommit to a future real option exercise choice but otherwise would not. We then show how a convertible bond can solve the precommitment problem. Section 4 introduces financing of a project in a static setting and Section 5 considers the problem of financing a multi-stage project. Section 6 concludes. Appendix A contains basic results from decision theory that are utilized in our analysis while Appendix B contains proofs for all propositions.

⁸See also Lopomo, Rigotti, and Shannon (2009) for an application of Knightian uncertainty to contract design and Lopomo, Rigotti, and Shannon (2011) for an application to moral hazard.

2 Ambiguity and real investment decisions

We consider a risk-neutral Entrepreneur (E) who has monopoly access to a project that requires an investment at time $t = 0$ and produces positive cash flows over two subsequent dates, $t = 1$, and $t = 2$. After the first cash flow is realized, E is able to either expand or contract the initial investment.

Capital markets: In this section we assume that E has sufficient funds to self finance and is able to affect output through the initial investment decision and subsequent expansion and contraction options. Despite having sufficient funds, however, we also consider the possibility that E may chose to contract with a Financier (F) in order to establish incentives needed to achieve particular real outcomes.

Ambiguity Defined: We study the affect of ambiguity on these real investment decisions and consider ambiguity to be in the mind of the decision maker (DM). The DM believes that a decision's stochastic outcomes are governed by a *set* of prior distributions. If an agent believes that the set of priors is a singleton, the situation is *risky* but *not ambiguous* and the decision maker follows the standard Subjective Expected Utility (SEU) criterion to determine his choice. We refer to an agent who sees a unique distribution governing outcomes as 'ambiguity neutral'. An investment decision is *ambiguous* if the agent believes that the set of priors contains more than one element and, in such cases, we will say that the agent is ambiguity averse.

2.1 Models of Decision Making In the Presence of Ambiguity

A number of models of decision making in the presence of ambiguity are found in the literature but we focus on only two: the Minimum Expected Utility (MEU) criterion, axiomatized by Gilboa and Schmeidler (1989), and the Consensus Expected Utility (CEU) criterion, axiomatized originally by Bewley (1986), and, more recently by Gilboa, Maccheroni, Marinacci, and Schmeidler (2010). As we will see, these two models of decision making in the presence of ambiguity can lead to potentially opposite implication with regard to real investment decisions.

Our decision to focus on only two models of decision making was based on a number of considerations. We feel that examining more than two approaches would add length and distraction without significantly adding to the insights of the model. We examine MEU because it

is perhaps the most widely used approach to dealing with ambiguity and because it can explain the Ellsberg paradox, the source of pervasive evidence of ambiguity aversion.

We consider CEU because we feel it is especially appropriate for applications in corporate finance. The good fit with corporate problems comes from the ‘unanimity’ interpretation first suggested by Bewely. That is, in addition to viewing this approach as describing a single decision maker with multiple priors, we suggest it is also descriptive of a group, such as a board of directors, who make decisions on a consensus basis. More on this below.

In this section we provide a brief description of the different types of preferences we consider. A more formal review of basic results from decision theory is provided in Appendix A. In Subsection 2.2 we provide details of the technology and characterize the solution of dynamic real investment problem in Subsection 2.3.

To clearly illustrate the difference between SEU, MEU and CEU, let us consider a project with outcomes in two possible future states of the world, ‘up’ and ‘down’. If E adheres to the SEU axioms, then (see Theorem 3 in Appendix A) he will make his decisions under the assumption that there is a *unique* (subjective) probability p characterizing the realization of state ‘up’. Alternatively, if E does not adhere to the SEU axioms, then under both MEU or CEU axioms, he will make his decisions under the assumption that there is a *set* Π of possible probabilities π characterizing the realization of state ‘up’ (see Theorems 4 and 5 in Appendix A).⁹ Let us denote the set Π as follows,

$$\Pi = \{\pi \in [0, 1] : p - \epsilon \leq \pi \leq p + \epsilon, \epsilon > 0\}, \quad (1)$$

in which ϵ captures the “degree” of ambiguity. The case of $\epsilon = 0$ corresponds to SEU, in which the set Π collapses to the unique prior p .

To analyze the difference among SEU, MEU and CEU we consider a choice between two “gambles” f and g , i.e., mappings from the state space S to the space of outcomes X . In what follows we consider the choice of a risk-neutral decision maker and adopt the notation $E_\pi[f]$ to indicate the expected value of gamble f under a specific prior distribution $\pi \in \Pi$.

⁹In the robustness literature it is common to refer to the probabilities π as to “models”, i.e., probabilistic description of a data generating process (see, e.g., Hansen and Sargent (2011)). In this literature, the concept of “prior” is a belief over a model (i.e., a second order belief). Because we do not deal with second order beliefs, in our setting the term “prior” is equivalent to the concept of “model” used in the robustness literature.

2.1.1 Subjective Expected Utility (SEU)

The preferences of a decision maker who adheres to Savage's (1954) axioms are represented via the well known *Subjective Expected Utility* (SEU) criterion (see Theorem 3 in Appendix A for a formal derivation), i.e., given any two gambles f and g ,

$$f \succ g \iff E_p[f] > E_p[g]. \quad (2)$$

The above representation makes clear that a SEU decision maker relies on a *unique* subjective prior p to choose between alternative gambles. In terms of the prior set (1), SEU corresponds therefore to the case in which $\epsilon = 0$.

Figure 1, Panel A, graphically illustrates the SEU criterion for the case of a two-dimensional state space $S = \{down, up\}$. A gamble f is identified by a pair of outcomes $(d, u) \in X = \mathbb{R}^2$. The downward sloping straight line through f represents the indifference curve for SEU, i.e., the set of gambles g such that $E_p[g] = E_p[f]$. The dark grey region represent the set of acts g preferred to f , $g \succ f$, while the light grey region represents the set of acts g that are dominated by f , i.e. $f \succ g$.

2.1.2 Minimum Expected Utility (MEU)

The preferences of a decision maker who adheres to Gilboa and Schmeidler's (1989) axioms, can be represented via the *Minimum Expected Utility* (MEU) criterion (see Theorem 4 in Appendix A for details). Specifically, for this DM, there exists a set of priors Π such that, given any two gambles f and g ,

$$f \succ g \iff \min_{\pi \in \Pi} E_{\pi}[f] > \min_{\pi \in \Pi} E_{\pi}[g]. \quad (3)$$

In the representation (3), the quantities $\min_{\pi \in \Pi} E_{\pi}[f]$ and $\min_{\pi \in \Pi} E_{\pi}[g]$ are the utility indexes for the gambles f and g respectively.

Figure 1, Panel B, provides a diagrammatic representation of MEU preferences for the case of a two-dimensional state space $S = \{down, up\}$. Using the set of priors (1) and the representation (3), we have that the utility index of the gamble $f = (d, u)$ is

$$\min_{\pi \in [p-\epsilon, p+\epsilon]} E_{\pi}[f] = \begin{cases} (p - \epsilon)u + (1 - p + \epsilon)d, & \text{if } u > d \\ (p + \epsilon)u + (1 - p - \epsilon)d, & \text{if } u < d \end{cases} \quad (4)$$

Hence, the MEU indifference curve through f is given by the kinked line in Panel B of Figure 1. Note that the kink in the indifference curve occurs at the point where $d = u$, along the 45 degree line. When $u > d$ the minimum expected utility is determined by the relatively pessimistic prior $\pi - \epsilon$ while, when $u < d$ the lowest expected utility is determined by the relatively optimistic prior $p + \epsilon$

For comparison, we also report the indifference curve through f of a SEU DM (dashed line). The dark grey region represent the set of acts g preferred to f , $g \succ f$, while the light grey region represents the set of acts g that are dominated by f , i.e. $f \succ g$. From the figure it is obvious that MEU preferences are complete.

2.1.3 Consensus Expected Utility (CEU)

In an attempt to provide a rigorous characterization of Knight’s (1921) distinction between risk and uncertainty, Bewley (2002) develops a theory of choice under uncertainty (ambiguity) that starts from Savage’s (1954) SEU axioms and drops the axiom of completeness. Once this axiom is removed, the decision maker’s preferences cannot be represented via a unique probability distribution. Instead, the choice between two gambles can be represented via a *Consensus Expected Utility (CEU)* criterion (see Theorem 4 in Appendix A for details). Under this “unanimity” criterion, a gamble f is preferred to a gamble g if its expected value is higher under *all* possible probability distributions in Π , i.e.,

$$f \succ g \iff E_\pi[f] > E_\pi[g], \forall \pi \in \Pi, \quad (5)$$

or, equivalently,

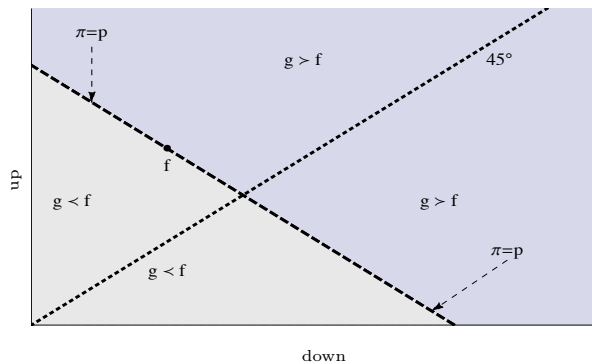
$$f \succ g \iff \min_{\pi \in \Pi} E_\pi[f - g] > 0. \quad (6)$$

Figure 1, Panel C provides a diagrammatic representation of CEU preferences in the case of a two-dimensional state space $S = \{\text{down}, \text{up}\}$. The dark grey region represents the set of gambles g that are preferred to f . According to (5), this region is the intersection of the “better-than- f ” sets of an SEU DM with prior $p - \epsilon$ and of an SEU DM with prior $p + \epsilon$. A similar observation can be made for gambles that are worse than f , represented by the light grey region in the figure. For comparison purposes we report also the indifference curve of the SEU DM (dashed line).

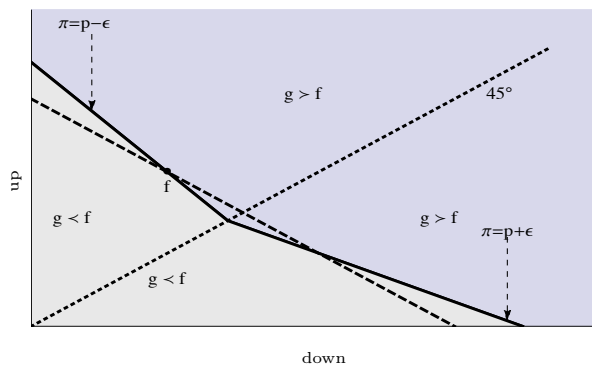
Figure 1: SEU, MEU and CEU preferences

The figure reports the indifference curve through the gamble f for SEU (Panel A) and MEU (Panel B). Panel C reports the preference ordering according to CEU. In all panels, the dark grey region indicates “better-than- f ” gambles and the light grey region indicates “worse-than- f ” gambles.

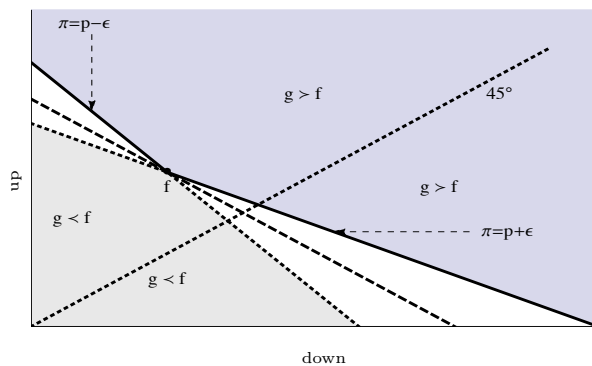
Panel A: SEU



Panel B: MEU



Panel C: CEU



Comparing Panels B and C, it is immediate to see the incomplete nature of the CEU ordering. The un-shaded regions in Panel C contain gambles that are not comparable to f . Unlike SEU and MEU, the CEU criterion does not assign a unique “value” to a gamble and therefore cannot always specify what the decision maker will do when faced with a choice. To resolve this indeterminacy Bewley introduces the following *inertia* assumption:

Inertia assumption. *A Decision Maker identifies one of the alternatives as the status quo and will only accept an alternative gamble if the expected value of the alternative is strictly better than that of the status quo under all possible priors.*

In the rest of this paper we will refer to CEU as to the unanimity criterion (5) *augmented* by the Inertia assumption. Therefore, assuming, for example, that f is the status quo in Panel C of Figure (1), and that g is a gamble in one of the un-shaded regions. Then $E_\pi[f] \not> E_\pi[g]$ and, as a consequence of the inertia assumption, the DM chooses to remain with the status quo f rather than moving to g .

2.1.4 CEU as a Model of Group Behavior

We believe that CEU is an appropriate description of group behavior where the group contains individuals who may hold unique or multiprior distributions over the outcomes of an action. Consider a board of directors considering an alternative to what they see as the status quo. Define Π^G as the union of the set of all sets of priors, Π^i of all individuals i in the group. If they will only accept alternatives if they are unanimously agreed to, they will behave as if they were the single agent CEU decision maker described above who holds the set of priors Π^G . This is because the alternative action will be evaluated relative to the status quo for all elements of the group’s set of priors.

This interpretation of CEU suggests many empirical questions that could be related to investment and financing decisions. For instance:

1. Where do the multiple priors come from? Insights into this question may be gleaned from the personal histories of the individuals. For instance, empirical behavioral finance studies have used the fact that an individual was alive during the great depression as a proxy for conservative preferences. An alternative is that these individuals have multiple priors, having seen a world that younger colleagues might feel is impossible.

2. How does a group define the status quo? For instance, when faced with the choice of continuing to pay fixed costs to continue producing or shutting down a factory, the group is not able to 'just do nothing'. Hence, the definition of the status quo is itself a group decision.
3. Do groups such as boards follow a unanimity rule? Although boards govern by majority votes, do they, as an empirical matter, only make decisions when there is unanimous agreement?

2.2 Technology

We now study how the above preferences impact the decisions of an entrepreneur E faced with a dynamic real investment problem. Specifically, we study decisions made by E at two different points in time. The first decision is whether or not to invest I_0 to acquire one unit of capital at $t = 0$. If the initial investment is made, E faces a second decision at $t = 1$ when the scale of the initial investment can be maintained, expanded or contracted.

The state of the world at time $t = 1, 2$ is represented by the cash flow s_t produced by one unit of capital. After the initial investment the project delivers cash flows C_t at times $t = 1$ and $t = 2$ that depend on the state of nature, s_t , and the scale of the firm, λ_t , according to

$$C_t = \lambda_t \times s_t. \quad (7)$$

The state of the world evolves over the two periods according to a simple binomial process. Starting at the unambiguous value s_0 , the process evolves to \tilde{s}_1 at time $t = 1$ where the random state \tilde{s}_1 takes the value s_u ('up state') or s_d ('down state'). Conditional on being in the 'up state' (resp. 'down state') at time $t = 1$, the state in the second period is denoted by $\tilde{s}_2 = \tilde{s}_2^u$ (resp. $\tilde{s}_2 = \tilde{s}_2^d$) where the \tilde{s}_2^u (resp. \tilde{s}_2^d) takes the value s_{uu} (resp. s_{du}) in the 'up-up state' (resp. 'down-up state') or s_{ud} (resp. s_{dd}) in the 'up-down state' (resp. 'down-down state'). We assume that

$$s_{ju} > s_{jd}, \quad j = u, d. \quad (8)$$

To simplify exposition, we assume that the same set Π describes conditional one-step-ahead priors at time $t = 0$ and at time $t = 1$ in both 'up' and 'down' states. This assumption essentially

imposes independence between successive realizations of the state.¹⁰ This in turns implies that, upon observing the realization of \tilde{s}_1 , E can only refine the set of future paths (binomial subtrees) but cannot learn more about the law of motion governing the random variable \tilde{s}_2 .¹¹

If the initial investment is made, the scale of the firm at $t = 1$ is $\lambda_1 = 1$ implying that $C_1 = s_1$. At $t = 1$ the entrepreneur can: (i) maintain the scale of the firm at 1 by expending m on maintenance; (ii) expand the firm by paying the amount I_1 at $t = 1$ to increase the scale of the firm to $\lambda_2 > 1$; or (iii) contract by scrapping the firm and receiving the scrap value of R at $t = 2$. The contraction choice correspond to selecting a scale $\lambda_2 = 0$ and receiving in return the scrap value R .

Figure 2: Expansion and contraction decisions

The figure illustrates the expansion and contraction decisions at time $t = 1$.

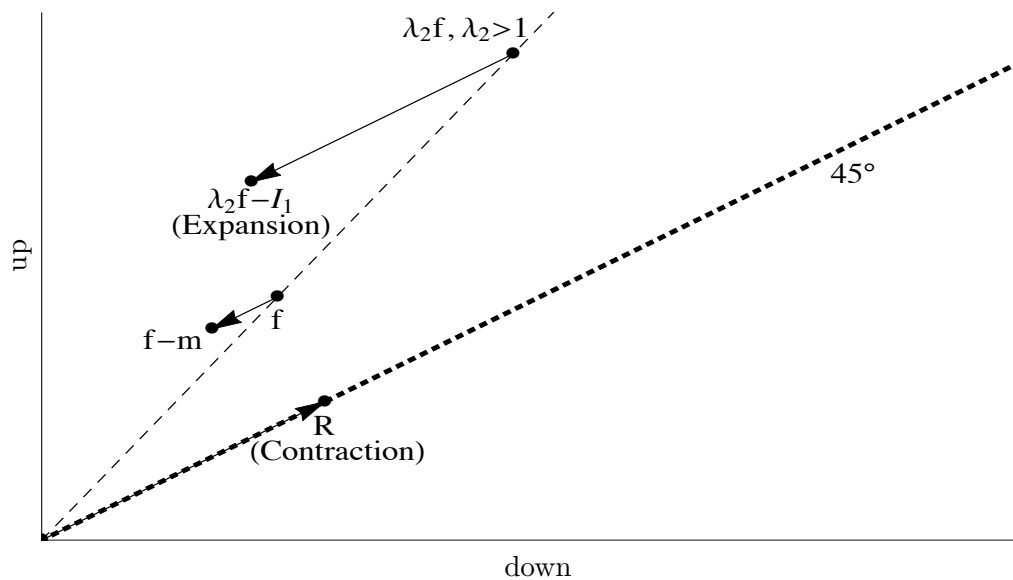


Figure 2 provides a diagrammatic representation of the choice available at time $t = 1$. To simplify notation, we denote by f the cash flow \tilde{s}_2^j , $j = u, d$, obtained at $t = 2$ if the entrepreneur continues operations at the scale $\lambda_2 = 1$. Because of assumption (8), f lies above the 45-degree

¹⁰A concrete way to understand our assumptions on ambiguity is to assume that nature flips a first coin at time $t = 1$ to determine the current shock (up or down). At time $t = 2$ a second independent coin is used to determine if the second shock is an up or a down movement. Both coins are ambiguous with a set of priors given in (1).

¹¹Because we take the one-step-ahead priors as primitive in our description of ambiguity, the resulting unconditional set of priors over the the state space satisfies by construction the rectangularity condition of Epstein and Schneider (2003) which guarantee time consistency of MEU preferences.

line. A convenient way to understand a change in scale (expansion or contraction) is to consider the ray going through f (dashed line in the figure).

The decision to expand at time $t = 1$ corresponds of moving upward along the ray from f to $\lambda_2 f$, $\lambda_2 > 1$, while the decision to contract corresponds to moving downward along the ray. If the entrepreneur decides to expand, it will have to pay an (unambiguous) cost I_1 to obtain a “scaled” up version of the current firm. The expansion decision therefore entails comparing the cash flow from current operations f , net of maintenance costs m , with the alternative $\lambda_2 f - I_1$. Expansion occurs at time $t = 1$ in state $j = u, d$ if

$$\lambda_2 f - I_1 \succ f - m. \quad (9)$$

By inequality (8) and the fact that the expansion cost I_1 is unambiguous, the “gamble” $\lambda_2 f - I_1$ lies always to the *left* of the ray going through f .¹² If the entrepreneur decides to contract, he selects a scale $\lambda_2 = 0$ and receives an (unambiguous) scrap value of R . Hence, the contraction decision entails comparing the cash flow from current operations f net of maintenance costs m with the alternative R on the 45-degree line. Contraction will occur at time $t = 1$ in state $j = u, d$ if

$$R \succ f - m. \quad (10)$$

By inequality (8) cash flows corresponding to contraction decisions will always lie to the *right* of the ray going through f .¹³

2.3 Recursive solution of the dynamic real investment problem

In this section we solve the investment decision problem recursively determining first the expansion/contraction decision at time $t = 1$ and then the initial investment decision at time $t = 0$. In our analysis we focus on the difference between the solutions obtained under SEU, MEU and CEU.

¹²The opposite is true, obviously, if inequality (8) were to be reversed.

¹³In general we could also consider a partial contraction, in which E selects a scale $\lambda_2 \in [0, 1)$ and receives a scrap value R in exchange of selling a fraction $1 - \lambda_2$ of the firm. Graphically this would entail comparing the cash flow from current operations $f - m$ with the alternative $\lambda_2 f + R$, $\lambda_2 \in [0, 1)$. Because the contraction payoff R is unambiguous, the gamble $\lambda_2 f + R$ lies always to the *right* of the ray going through f .

In the context of dynamic models of choice under ambiguity, an issue that frequently arises is that of dynamic consistency. It is well known¹⁴ that, under a general information structure, MEU preferences can lead to time inconsistent choices. As we discussed in footnote 11, our information structure is built by taking the one-step-ahead priors as primitive in our description of ambiguity. As shown by Epstein and Schneider (2003), the resulting *unconditional* set of priors over the the state space will satisfies the “rectangularity condition” that guarantees time consistency of MEU preferences. However, as we show in Section ??, CEU preferences can exhibit dynamic inconsistency because of the consequence of change of status quo over time.

2.3.1 Time 1 decisions

Figure 3, Panel A, describes the expansion/contraction choice under MEU preferences at time $t = 1$ and compares it to SEU. The figure combines the preference description in Figure 1, Panels A and B, with the technology description in Figure 2. The point f represents the cash flow of one unit of capital at time $t = 2$ (continuation) under either state ‘up’ or ‘down’. The kinked line is the MEU indifference curve through $f - m$, while the dotted line is the indifference curve for a SEU DM. The SEU decision is simply

$$\text{Expand} \succ_{\text{SEU}} \text{Continue} \iff \lambda_2 E_p[f] - I_1 > E_p[f] - m \quad (11)$$

$$\text{Contract} \succ_{\text{SEU}} \text{Continue} \iff R > E_p[f] - m, \quad (12)$$

where the DM makes use of the unique probability p in assessing alternatives.

From the figure we immediately see that an MEU DM will use the prior $\pi = p - \epsilon$ to decide whether to expand or contract. There is a sufficiently high level of investment cost I_1 and a sufficiently low level of maintenance cost m for which and MEU DM will not expand while an SEU DM will. Similarly, there are scrap values R for which MEU will contract while an SEU will not. The choice of a MEU DM are then

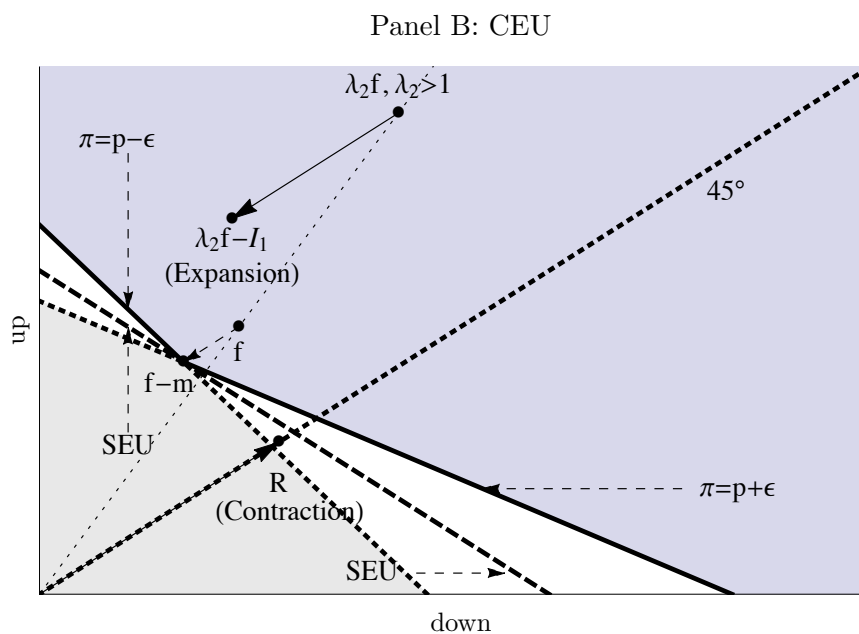
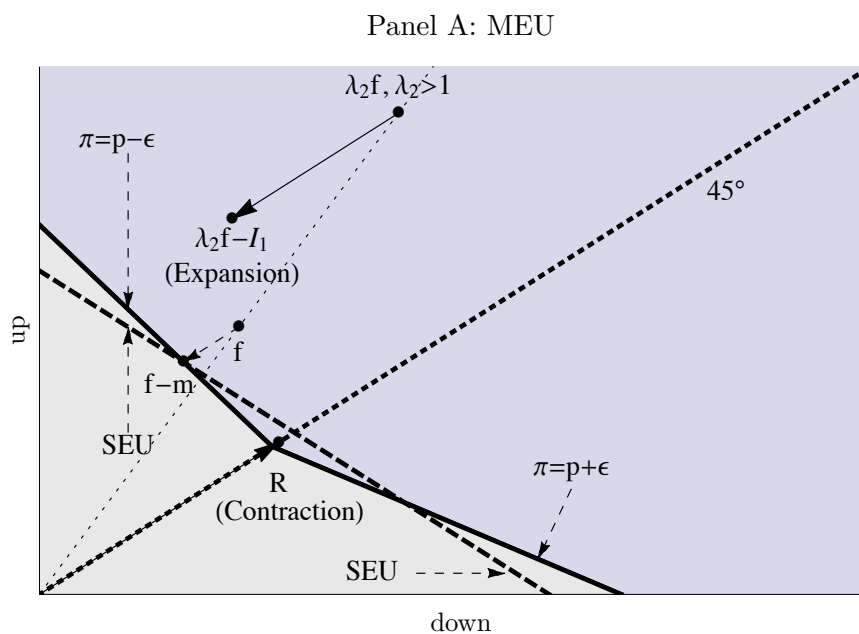
$$\text{Expand} \succ_{\text{MEU}} \text{Continue} \iff \lambda_2 E_{p-\epsilon}[f] - I_1 > E_{p-\epsilon}[f] - m \quad (13)$$

$$\text{Contract} \succ_{\text{MEU}} \text{Continue} \iff R > E_{p-\epsilon}[f] - m, \quad (14)$$

¹⁴See, for example, Al-Najjar and Weinstein (2009) and Siniscalchi (2011) for an analysis of the time inconsistency issue in dynamic choice under ambiguity.

Figure 3: Time 1 decisions

Panel A (B) reports the expansion/contraction decision under MEU (CEU) preferences. In both panels, the dotted line refers to the indifference curve under SEU preferences.



In summary, the analysis in Figure 3, Panel A, shows that for the expansion and contraction decisions, the MEU DM behaves as a pessimistic SEU who has beliefs captured by the prior $p - \epsilon$.

Panel B of Figure 3 illustrates the key difference between MEU and CEU preferences. As in the MEU case the relevant prior that determines whether an expansion decision is taken or not is $\pi = p - \epsilon$. However, the relevant prior for deciding whether to contract or not is now $\pi = p + \epsilon$ and no longer $\pi = p - \epsilon$, as it was for the case of MEU preferences. When offered the option to sell an ambiguous firm, the CEU DM considers the best possible scenario as a criterion to decide whether to surrender the firm for an unambiguous amount of cash R . The choice of a CEU DM are then

$$\text{Expand} \succ_{\text{CEU}} \text{Continue} \iff \lambda_2 E_{p-\epsilon}[f] - I_1 > E_{p-\epsilon}[f] - m \quad (15)$$

$$\text{Contract} \succ_{\text{CEU}} \text{Continue} \iff R > E_{p+\epsilon}[f] - m, \quad (16)$$

The figure also shows that both contraction and expansion are less “likely” for a CEU DM than for an SEU DM, whose indifference curve through $f - m$ is represented by the dashed straight line.

In summary, the analysis in Figure 3 shows that while it is difficult to distinguish SEU from MEU and CEU for expansion decisions beyond relative pessimism, a contraction decision delivers very different results. While the MEU contraction decision (14) is indistinguishable from SEU contraction (12) with a *unique* “pessimistic” prior $\pi = p - \epsilon$, the CEU decision is clearly different from both SEU and MEU. A comparison of (14) with (16) shows that when facing a contraction an MEU decision maker is pessimistic while a CEU decision maker is optimistic. Hence, even when E can sell a firm for R and may even consider R to be larger than his expected payoff from continuing under some priors, he will *not* contract if he entertains priors under which the expected continuation value of the firm is higher than R .

It has been observed that managers are reluctant to divest or shut down projects that have not done well, a result that has been explained by agency problems, reputation concerns and asymmetric information (see, for example, Boot (1992) and Weisbach (1995)). CEU can provide an alternative explanation in terms of ambiguity even in the absence of asymmetric information.

If DMs are ambiguity sensitive and adhere to the CEU axioms they will be reluctant to terminate a project because of the importance that potential regret plays in their decision making.

Finally, note that there is nothing special with the connection “contraction–optimism ($p + \epsilon$)” and “expansion–pessimism ($p - \epsilon$)”. Indeed these connections will be reversed in the case inequality (8) were to hold with the opposite sign and the status quo cash flows were higher in state ‘down’ than in ‘up’.

2.3.2 Time 0 decision

At time $t = 0$ the entrepreneur is endowed with an unambiguous amount of funds I_0 . His choice is whether to keep I_0 or acquire one unit of capital which, in turn, will give him the opportunity to expand or contract as discussed above. To obtain the recursive solution, we take the expansion and contraction decisions as given and use this information to determine E 's choice at time $t = 0$.

While it is possible to analyze the investment decision in generality, we want to focus on the cases in which the solution under MEU and CEU differ at time $t = 0$. Choosing the status quo I_0 at time $t = 0$ results in an unambiguous “gamble” at time $t = 1$ with payoff equal to I_0 in each state ‘up’ and ‘down’. Using the preference representation in Figure 1, the choice “not invest” will therefore be a point on the 45-degree line. From Panel B and C of Figure 1 we then notice that the indifference curve of an MEU DM through I_0 corresponds to the contour set of the “better than I_0 ” gambles for a CEU DM. Moreover, because I_0 is the status quo, whenever $g \neq I_0$, a CEU DM will stick with the status quo I_0 . This implies that, when the status quo is I_0 , because of the implicit inertia assumption, if faced with the *same* gambles at time $t = 1$, CEU preferences will *de facto* deliver the *same* preference ordering as MEU preferences. Hence, any difference between time $t = 0$ decisions of MEU and CEU can only come from their disagreement in decisions at time $t = 1$.

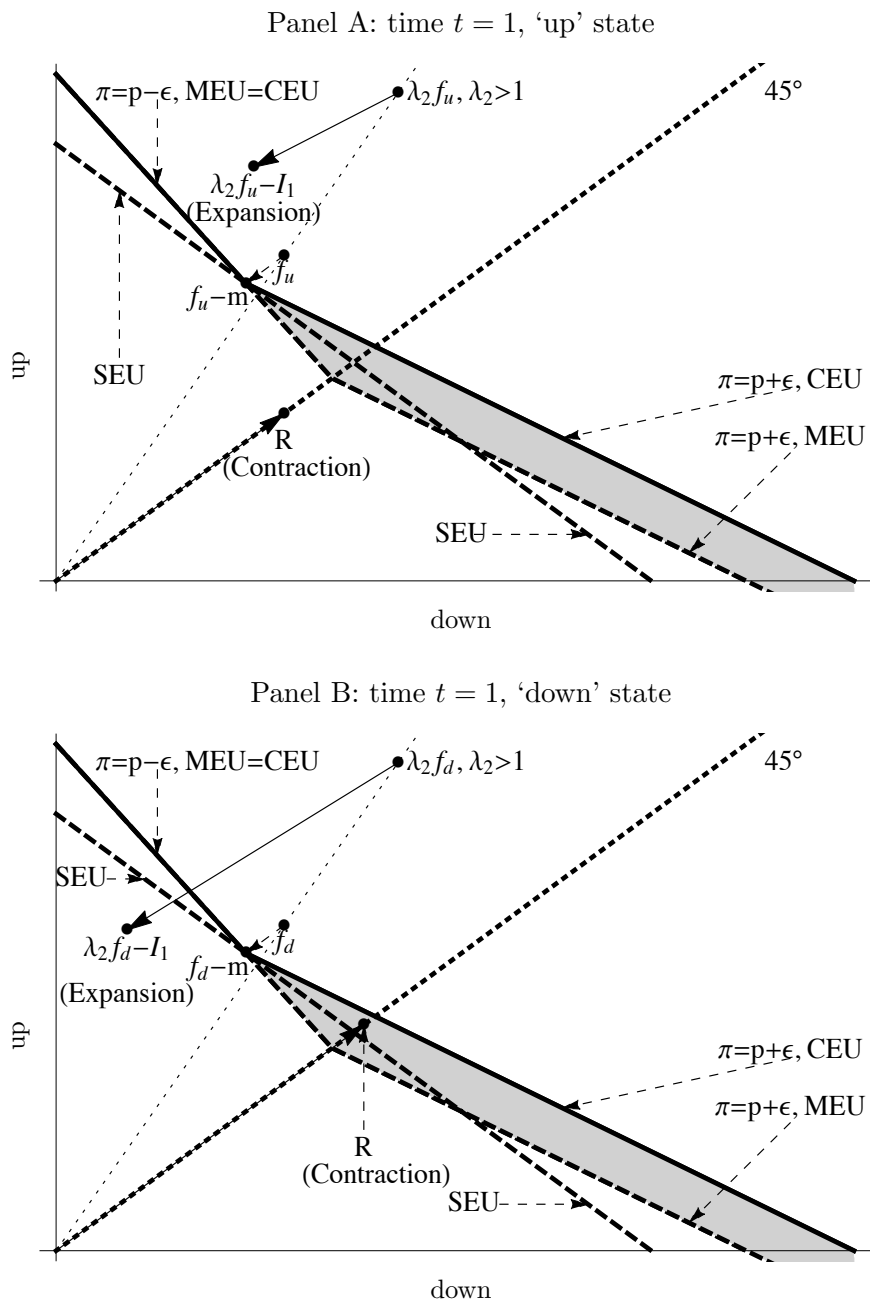
To illustrate the potential difference in the dynamic investment decision under SEU, MEU and CEU, let us consider the following example. Suppose that for the decision problem at time $t = 1$ in state ‘down’,

$$\text{Expansion decision:} \quad \lambda_2 E_p[\tilde{s}_2^d] - I_1 < E_p[\tilde{s}_2^d] - m \quad (17)$$

$$\text{Contraction decision:} \quad E_{p-\epsilon}[\tilde{s}_2^d] < E_p[\tilde{s}_2^d] < R < E_{p+\epsilon}[\tilde{s}_2^d], \quad (18)$$

Figure 4: Time 1 decisions: MEU vs CEU

The shaded area in each panel represents the set of gambles over which MEU and CEU will disagree. The dotted line represents the SEU indifference curve through $f - m$.



and for the decision problem at time $t = 1$ in state ‘up’,

$$\text{Expansion decision: } \lambda_2 E_{p-\epsilon}[\tilde{s}_2^u] - I_1 > E_{p-\epsilon}[\tilde{s}_2^u] \quad (19)$$

$$\text{Contraction decision: } R < E_{p-\epsilon}[\tilde{s}_2^u], \quad (20)$$

Conditions (17)–(18) imply that, in the ‘down’ state state expansion is not undertaken under any preference, while contraction is undertaken by SEU and MEU but not by CEU (see Panel A of Figure 4). Conditions (19)–(20) imply that, in the ‘up’ state, expansion is undertaken under all preference criteria while contraction is a dominated choice for all preferences (see Panel B of Figure 4).

Because SEU and MEU are complete order, we can use the value the DM attaches to his choice at time $t = 1$ to determine the optimal choice at time $t = 0$. The SEU value of expanding in state ‘up’ is $\lambda_2 E_p[\tilde{s}_2^u] - I_1$ while the value of contracting in state ‘down’ R . Hence the SEU decision at time $t = 0$ is

$$\text{Invest } \succ_{\text{SEU}} \text{ Don't invest } \iff p(\lambda_2 E_p[\tilde{s}_2^u] - I_1) + (1-p)R > I_0. \quad (21)$$

Similarly, the MEU value of expanding in state ‘up’ is $\lambda_2 E_{p-\epsilon}[s_2^u] - I_1$ while the value of contracting in state ‘down’ R . Notice that, because of inequalities (19) and (20), $\lambda_2 E_{p-\epsilon}[s_2^u] - I_1 > R$ and hence the MEU decision at time $t = 0$ is

$$\text{Invest } \succ_{\text{MEU}} \text{ Don't invest } \iff (p-\epsilon)(\lambda_2 E_{p-\epsilon}[s_2^u] - I_1) + (1-p+\epsilon)R > I_0. \quad (22)$$

As discussed above, CEU preferences are not complete, and hence we cannot assign a unique “value” to a particular gamble. This makes the procedure we followed for SEU and MEU infeasible for CEU. To solve for the time $t = 0$ decision under CEU preferences, we invoke Strotz’s (1955) “consistent planning” criterion and solve the decision problem of a probabilistically sophisticated DM that anticipates the future contraction/expansion choices he will make. The CEU decision at time $t = 0$ involves solving a *static* optimization problem in which the DM compares I_0 to the ambiguous payoff emerging at time $t = 2$ from following the decisions dictated by the conditions (17) and (20) above, i.e., expanding in the ‘up’ state and continuing in the ‘down’ state.

To formalize the decision problem of CEU at time $t = 0$, let $\pi_0 \in \Pi$ denote the one-step ahead prior at time $t = 0$, and $\pi_1^u, \pi_1^d \in \Pi$ the conditional one-step ahead priors at time $t = 1$. Because of independence, the unconditional probabilities of states of the world at time $t = 2$ are the product of one-step ahead priors. In general, therefore, the CEU assessment at time $t = 0$ of the investment plan f implied by conditions (17)–(20) can be characterized as follows:

$$\text{Invest} \succ_{\text{CEU}} \text{Don't invest} \iff \min_{\pi_0, \pi_1^u, \pi_1^d \in \Pi} E_{\pi_0, \pi_1^u, \pi_1^d}[f] > I_0, \quad (23)$$

where

$$E_{\pi_0, \pi_1^u, \pi_1^d}[f] = \pi_0 E_{\pi_1^u}[\lambda_2 \tilde{s}_2^u - I_1] + (1 - \pi_0) E_{\pi_1^d}[\tilde{s}_2^d]. \quad (24)$$

In Proposition 8 of Appendix B we show that to determine the minimum in (24) it is sufficient to analyze the corner of the set Π^3 . , we have that

$$\text{Invest} \succ_{\text{CEU}} \text{Don't invest} \iff (p - \epsilon)(\lambda_2 E_{p-\epsilon}[\tilde{s}_2^u] - I_1) + (1 - p + \epsilon) E_{p-\epsilon}[\tilde{s}_2^d] > I_0. \quad (25)$$

Comparing (22) with (25) we see that, under condition (18) above, $R > E_p[\tilde{s}_2^d] > E_{p-\epsilon}[\tilde{s}_2^d]$ and therefore, the impact of future delays in contractions by the CEU entrepreneur is to adopt a more conservative investment policy than that of a MEU entrepreneur.

3 Commitment and real investment decisions

A somewhat undesirable property of CEU preferences with inertia is that, in a dynamic choice problem, decisions can be *dynamically inconsistent*. This happens when the choice a DM is willing to *precommit* to as of time zero does not coincide with the choice the same DM will make when solving his problem recursively via backward induction (see Strotz (1955)). In other words, dynamic inconsistency in preferences arises when a ranking is reversed by the mere passage of time.¹⁵

In the context of our real investment problem, dynamic inconsistency can arise as a consequence of the change in status quo over time. The analysis in the previous section shows that the decision maker faces a tension between his time-0 self and his time-1 self: the time-0 self forgoes investment opportunities *because* of the inertia that the time-1 self exhibit. Were he able

¹⁵As discussed above, dynamic inconsistency can arise also with MEU preferences if the set of priors is not rectangular (see Al-Najjar and Weinstein (2009) and Siniscalchi (2011)).

to precommit to time-1 decisions, the time-0 self would be willing to undertake the investment project.

In this section we illustrate how the inertia induced by CEU preferences, and the resulting dynamically inconsistent behavior, can be partially overcome through the use of securities issued or purchased by the entrepreneur at the time of the initial investment decision. Specifically, in the example of the previous section, we show that securities that have convertible features, such as convertible bonds, can provide the correct incentives to shut down firms in bad time, thus making the investment decision attractive as of time zero.

Note that the conservative behavior exhibited by individuals with CEU preference, and the resulting time inconsistency problem, can equally emerge in the context of *group* decision making where group members have different beliefs and where decisions are taken under a unanimity rule. The analysis of CEU preferences addresses therefore a more general spectrum of situation than those attributed to a single decision maker. For the rest of the paper we will refer to a single decision maker but it should be understood that all the analysis related to CEU preferences applies unchanged to groups that decide according to a unanimity rule.

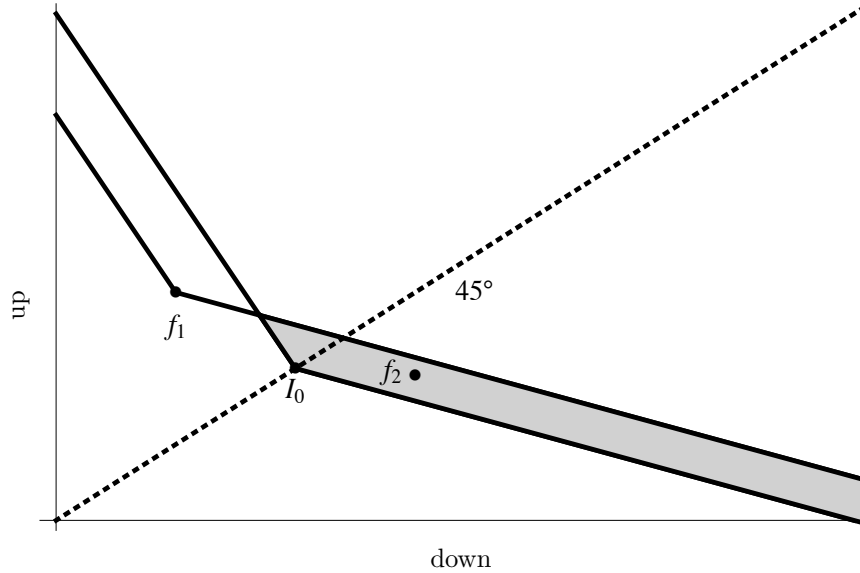
3.1 Dynamic inconsistency of CEU preferences: general principles

To see how dynamic inconsistency can arise in the context of our real investment problem, consider a generic two-period, three-dates problem in which the DM is asked to make a decision at time $t = 0$ and at time $t = 1$. Each decision involves comparing gambles on a two-dimensional state space, as those described in the previous sections. Let us assume, without loss of generality, that the status quo at time $t = 0$ is an unambiguous gamble I_0 . Figure 5 reports I_0 on the 45-degree line. The solid line with a kink at I_0 represents the contour of the set containing gambles that are preferred to I_0 . In the figure we also report the gambles f_1 , which represents the status quo at time $t = 1$. In summary, at time $t = 0$ the DM compares I_0 vs. f_1 and at time $t = 1$, the comparison is between f_1 and f_2 .

The shaded area in the figures denotes the regions of gambles with $t = 2$ payoffs that will give rise to dynamically inconsistent choices. To see this, consider first the *recursive* solution. To obtain this solution let us start from the decision at time $t = 1$ and compare gambles f_1 and f_2 . Because $f_2 \succ f_1$, the DM at time $t = 1$ will stick with the status quo f_1 . Given this, the DM

Figure 5: Dynamic inconsistency of CEU preferences

The highlighted region in the figure refers to acts at time 2 that will lead to disagreement between commitment and recursive solution.



at time $t = 0$ will then compare I_0 with f_1 . Because $f_1 \not\succeq I_0$ the time-0 DM will stick with the status quo I_0 . Hence I_0 is the *recursive solution* for the dynamic problem portrayed in Figure 5.

Let us now consider the *commitment* solution. Under pre-commitment, the DM at time 0 decides whether he would like to “precommit” to gamble f_1 and/or to f_2 . Because $f_1 \not\succeq I_0$, f_1 cannot be a pre-commitment choice. However $f_2 \succ I_0$. Notice that, because f_2 can be obtained only from f_1 the precommitment choice involves an action of the type “ f_1 and then f_2 ”. Hence the *pre-commitment solution* involves choosing f_1 at time $t = 0$ and then f_2 at time $t = 1$. As shown above, this choice cannot be implemented recursively: Pre-commitment and recursive solution disagree. It is possible to show that in a dynamic problem involving the three gambles I_0 , f_1 and f_2 , dynamic inconsistency arises when: (i) $f_1 \not\succeq I_0$ and (ii) $f_2 \not\succeq f_1$. Under these conditions there is a change in status quo over time and the two status quibus cannot be ranked. The shaded region in Figure 5 illustrates the set of gambles f_2 for which dynamic inconsistency arises.

Figure 5 also helps us understand how contracts can help resolve the dynamic inconsistency problem. By signing contracts at time 0, the time-0 DM can affect the payoff of the gamble f_2 . A contract can resolve inconsistency if it can change the payoff of either f_1 and/or f_2 , so that

the new gambles f'_1 and f'_2 can be ranked. In this case, f'_2 turns out to be both the commitment and recursive solution.

In the rest of this section we illustrate these principles in the context of the real investment decision discussed in Section 2 and show how derivative contracts can be useful to tools for the “time-0” entrepreneur to discipline the decisions of his “time-1” self. Because SEU and MEU preferences are time-consistent under our information structure, we focus the rest of the section on CEU preferences.

3.2 Dynamic inconsistency: a contractual solution

Let us consider the two-period real investment problem discussed in Section 2 where, for simplicity, we ignore the maintenance cost m from continuing operation of an existing venture. Let us assume that the recovery value R and expansion cost I_1 satisfy the following conditions:

Assumption 1. *The recovery value R for shutting down the firm at $t = 1$ is such that*

$$E_{p-\epsilon}[\tilde{s}_2^d] < R < E_{p+\epsilon}[\tilde{s}_2^d]. \quad (26)$$

The investment cost I_1 for expanding the scale to λ_2 at time $t = 1$ is such that

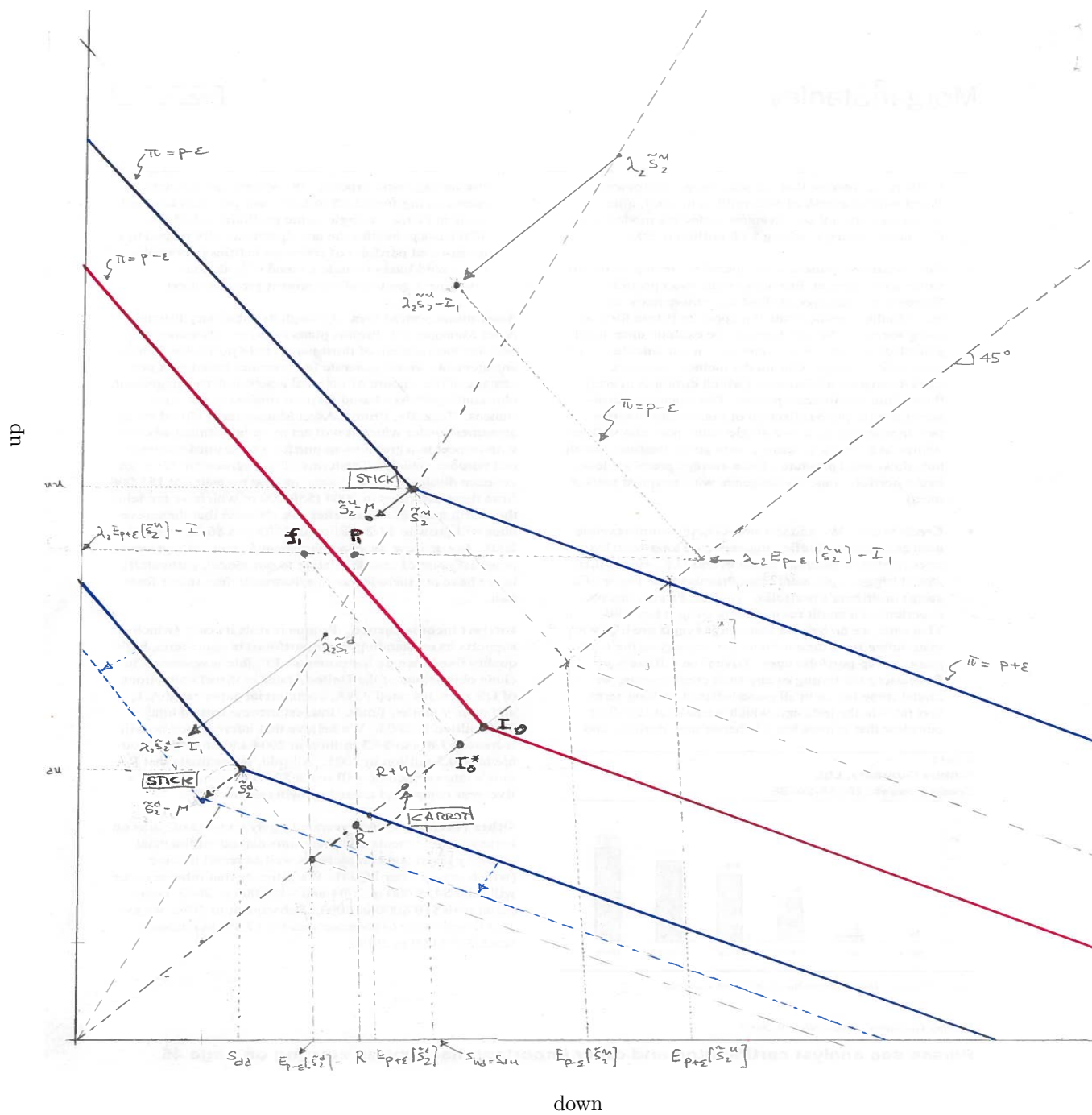
$$\lambda_2 E_{p-\epsilon}[\tilde{s}_2^d] - I_1 < E_{p-\epsilon}[\tilde{s}_2^d] < E_{p-\epsilon}[\tilde{s}_2^u] < \lambda_2 E_{p-\epsilon}[\tilde{s}_2^u] - I_1 \quad (27)$$

Assumption 1, guarantees that the entrepreneur expands in the up states and maintains the firm running in the down state. Figure 6 provides a diagrammatic illustration of the above conditions. Specifically, in the down state, (26) implies that the status quo is chosen over shutting down the firm and (27) implies that the status quo is chosen over expanding the firm’s scale. In the up state, conditions (27) and (26) imply that expansion is the preferred to the status quo while shutting down is always a dominated alternative.

The following proposition characterizes the recursive solution of the investment problem when the technology parameters satisfy Assumption 1 and provides conditions under which dynamic consistency can be violated, i.e., a pre-commitment solution differs from the recursive one.

Figure 6: Recursive and precommitment investment strategies

The figures illustrates the effect of stick- and carrot-like contracts on real investment decision in a dynamic model.



Proposition 1. *Suppose Assumption 1 is satisfied. Then there exist a cutoff level I_0^* for the time $t = 0$ investment cost such that the project is undertaken if $I_0 < I_0^*$, where*

$$I_0^* \equiv (p - \epsilon)(\lambda_2 E_{p-\epsilon}[\tilde{s}_2^u] - I_1) + (1 - p + \epsilon)E_{p-\epsilon}[\tilde{s}_2^d]. \quad (28)$$

If the project is undertaken, at time $t = 1$ the entrepreneur expands in the up state and keeps the status quo operating project in the down state.

Proof: The decision at time $t = 1$ follows directly from Assumption 1. The decision at time $t = 0$ is obtained by backward induction, taking as given the decisions at time $t = 1$. Because of conditions (26) and (27), $\lambda_2 E_{p-\epsilon}[\tilde{s}_2^u] - I_1 > E_{p-\epsilon}[\tilde{s}_2^d]$ and therefore the more stringent prior for a CEU DM is $p - \epsilon$. This justifies the cutoff I_0^* in (28). ■

Figure 6 provides a diagrammatic illustration of the proposition. In the figure $I_0 > I_0^*$ and the investment is not undertaken at time $t = 0$. The CEU entrepreneur is not investing because he anticipates that he will not be shutting down the firm at $t = 1$ in the down state.

Suppose the entrepreneur were able to choose between the two following *precommitment* strategies:

$$I_0 \quad : \quad \text{do not invest at time } t = 0 \quad (29)$$

$$P_1 \quad : \quad \text{invest at time } t = 0, \text{ expand in state "up", contract in state "down"}. \quad (30)$$

Strategy I_0 corresponds to the recursive solution obtained in Proposition 1 when $I_0 > I_0^*$ and is represented by the “gamble” I_0 in Figure 6. Strategy P_1 is a gamble with an ambiguous payoff of $\lambda_2 E_{p-\epsilon}[\tilde{s}_2^u] - I_1$, in the up state and R in the down state. From Figure 6 we see that the gamble $P_1 \succ I_0$. Because I_0 corresponds to the recursive solution when $I_0 > I_0^*$, we have that the precommitment strategy P_1 does not coincide with the recursive strategy I_0 .

Figure 6 also makes it clear that, in order for a strategy like P_1 to be obtained via a recursive solution to the investment problem, it is necessary to break the condition $I_0 > I_0^*$. For this to happen it is necessary to either (i) make the shut down option more attractive (“carrot”) by increasing the salvage value R , or (ii) make the continuation option less attractive (“stick”), by reducing the payoff from continuation \tilde{s}_2^d .

In the next two subsections we illustrate how by buying or issuing securities at time $t = 0$ the entrepreneur is able to affect the project’s payoffs in future states of the world and, consequently,

can “manipulate” the preferences of his future self in order to achieve payoffs profiles that, in the absence of these securities, can only be achieved via pre-commitment.

In Section 3.2.1 we consider pure “sticks-and-carrot” types of securities whose payoff are dependent on the action undertaken in the future. While purely theoretical, the study of these securities help understand the key forces that are required for a more general contract to be able to achieve the desired precommitment strategy. In Section 3.2.2 we show indeed that derivative contracts like convertible bonds are natural ways to implement the pure securities with action-dependent payoffs.

3.2.1 Precommitment via securities with action-dependent payoff

The difference between pre-commitment (P_1) and recursive (I_0) solutions highlighted in Figure 6 arises because the entrepreneur at time $t = 1$ does not have incentive to shut down the firm in the down state.

Let us suppose the entrepreneur can acquire a security g at time $t = 0$ that pays off $\eta > 0$ if he shuts down the firm. From Figure 6, we see that adding this security to the entrepreneur’s portfolio has the effect of increasing the recovery value of shutting down from R to $R + \eta$. If

$$E_{p+c}[\tilde{s}_2^d] < R + \eta < E_{p+c}[\tilde{s}_2^u], \quad (31)$$

then the entrepreneur will decide to abandon the status quo in the down state and shut down the firm (first inequality in (31)). The second inequality (31) guarantees that η is not too large to induce the entrepreneur to shut down also in the up state.

To see whether this carrot-like contract achieves the desired purpose of generating a recursive solution aligned with the precommitment solution P_1 in (30) we need to account for the cost the entrepreneur incurs at time $t = 0$ to acquire such a security. To this purpose, we assume that the financier who sells the security to the entrepreneur also has CEU preferences with the same prior set Π as the entrepreneur.¹⁶ Let P be the price charged by the financier to sell the carrot-like security that pays $\eta > 0$ in the event of the entrepreneur shutting down the firm. Under condition (31) the entrepreneur shuts down only in the down state and therefore the security g is a gamble with payoff 0 in the up state and $\eta > 0$ in the down state. The price P a CEU financier charges for issuing the security g must satisfy

¹⁶The analysis will be qualitatively similar if we were to assume a financier with SEU or MEU preferences.

$$P - g \succ 0 \iff P > E_\pi[g], \forall \pi \in \Pi, \text{ i.e., } P > (1 - p - \epsilon)\eta, \quad (32)$$

where we assumed a status quo of 0 for the financier. The entrepreneur pays the price P to acquire a security g that will induce him to expand in the up state and shut down in the down state. The investment is undertaken at time zero if

$$I_0 + P < (p - \epsilon)(\lambda_2 E_{p-\epsilon}[\tilde{s}_2^u] - I_1) + (1 - p - \epsilon)(R + \eta). \quad (33)$$

Because by (32) $P > (1 - p - \epsilon)\eta$, (33) is equivalent to

$$I_0 < (p - \epsilon)(\lambda_2 E_{p-\epsilon}[\tilde{s}_2^u] - I_1) + (1 - p - \epsilon)R. \quad (34)$$

The right hand side of (34) is exactly the payoff of the precommitment strategy P_1 which we assumed to be strictly preferred to the status quo I_0 . Therefore, by purchasing the carrot-like security g that pays off η when the firm is shut down the entrepreneur solves his commitment problem and eliminates the disagreement between recursive and precommitment strategies.

A similar outcome can be obtained via a “stick-like” security. Suppose instead that the entrepreneur issues at time $t = 0$ a security h according to which the issuer promises to pay an amount $\mu > 0$ in case the firm is *not* shut down. We can think of the payoff required by h as of a sort of maintenance cost. In Figure 6 the effect of this security is to shift the fulcra of the contour sets \tilde{s}_2^d and \tilde{s}_2^u to the points $\tilde{s}_2^d - \mu$ and $\tilde{s}_2^u - \mu$, respectively. Intuitively, the goal of this security is to make continuation in the down state less attractive and induce the entrepreneur to shut down. For this to be the case, the payoff μ has to be such that

$$E_{p+\epsilon}[\tilde{s}_2^d] - \mu < R < E_{p+\epsilon}[\tilde{s}_2^u] - \mu, \quad (35)$$

where the second equality guarantees that the penalty μ is not too large to make shutting down attractive also in the up state.

To see whether the stick-like contract h achieves purpose of generating a recursive solution aligned with the precommitment solution P_1 in (30) we need to account for the amount the entrepreneur raises at time $t = 0$ for issuing such a security. As before we assume that the financier who buys the entrepreneur’s promise also has CEU preferences with the same prior set Π as the entrepreneur. Let P be the price charged by the entrepreneur to issue the stick-like

security g . This security pays $\mu > 0$ when the entrepreneur keeps the firm operating or expands it at time $t = 1$. Under condition (35) and Assumption 1, the entrepreneur shuts down in the down state and expand in the up state. Hence the security h is a gamble with payoff $\mu > 0$ in the up state and 0 in the down state. The price P a CEU financier is willing to pay for purchasing the security h must satisfy

$$h - P \succ 0 \iff E_\pi[h] > 0, \forall \pi \in \Pi, \text{ i.e., } P < (p - \epsilon)\mu, \quad (36)$$

where we assumed a status quo of 0 for the financier. The entrepreneur receives the price P to issue a security h that will induce him to expand in the up state and shut down in the down state. The investment is undertaken at time zero if

$$I_0 - P < (p - \epsilon)(\lambda_2 E_{p-\epsilon}[\tilde{s}_2^u] - I_1 - \mu) + (1 - p - \epsilon)R. \quad (37)$$

Because by (36) $P < (p - \epsilon)\mu$, (37) is equivalent to

$$I_0 < (p - \epsilon)(\lambda_2 E_{p-\epsilon}[\tilde{s}_2^u] - I_1) + (1 - p - \epsilon)R, \quad (38)$$

where, again, the right hand side of (38) is exactly the payoff of the precommitment strategy P_1 which we assumed to be strictly preferred to the status quo I_0 . Therefore, by issuing the stick-like security h that promises to pay μ when the firm is not shut down the entrepreneur solves his commitment problem and eliminates the disagreement between recursive and precommitment strategies.

3.2.2 Precommitment via derivative securities

In this section we show that, by financing part of the project through properly designed convertible bonds, the entrepreneur can insure that his dynamic investment strategy coincides with the commitment strategy P_1 described in (30), i.e., the entrepreneur starts the project at time $t = 0$, expands in the up state and contracts in the down state.

Let us consider a convertible bond with face value X and a conversion ratio $0 \leq \alpha \leq 1$. We assume that the terms of the contracts are such that the bondholder can decide to convert the bond into α units of the firm at time $t = 2$, after the entrepreneur has decided whether to continue, expand or shut down. If we denote by f the payoff from the project at time $t = 2$,

then the bondholder payoff at maturity is

$$B_2 = \max\{\alpha f, X\} \quad (39)$$

where $f = R$ if the project is shut down at time $t = 1$, $f = \tilde{s}_2^i$, $i = u, d$ if the project is kept at the initial scale at time $t = 1$ and $f = \lambda_2 \tilde{s}_2^i - I_1$, $i = u, d$, if the entrepreneur decides to increase the scale of operation at time $t = 1$ in state $i = u, d$.

Let us assume that the face value X of the bond and the conversion ratio α are such that

$$\alpha s_{dd} < X < \min\{s_{dd}, \alpha R\} \quad (40)$$

Because, by Assumption 1, $R < s_{ud}$, condition (40) implies that the bond is risk-free, and that it will be converted in all the states at time $t = 2$ except the state in which the payoff from one unit of installed capital is s_{dd} . The bond payoff is therefore

$$B_2^{\text{expand in u}} = \alpha(\lambda_2 \tilde{s}_2^u - I_1) \quad (41)$$

$$B_2^{\text{continue in d}} = \begin{cases} \alpha s_{du} & \text{if } \tilde{s}_2^d = s_{du} \\ X & \text{if } \tilde{s}_2^d = s_{dd} \end{cases} . \quad (42)$$

If the entrepreneur instead shuts down the firm in the state down, because of condition (40), the payoff of the convertible bond will be

$$B_2^{\text{shut down in d}} = \alpha R \quad (43)$$

The payoff to the entrepreneur from owning the project f and issuing convertible bond is given by

$$f - B_2 = \min\{(1 - \alpha)f, f - X\} \quad (44)$$

Because of Assumption 1, in the absence of a convertible bond, shutting down the firm in state down does not dominate the status quo \tilde{s}_2^d , i.e., $\tilde{s}_2^d \neq R$. The presence of a convertible bond, translates the status quo of the down state from \tilde{s}_2^d to $\tilde{s}_2^d - \tilde{\mu}$, a gamble with payoffs $(1 - \alpha)s_{du}$ in the up state and $s_{dd} - X > 0$ in the down state. From Figure 6 we see therefore that the convertible bond is a special case of the “stick-like” contract with a state-dependent penalty $\tilde{\mu}$ equal to αs_{du} in the up state and X in the down state.

Furthermore, from (44) the convertible bond changes the payoff of the shut down option from R to $(1 - \alpha)R$. This payoff is of the “carrot-like” form $R + \eta$, where $\eta = -\alpha R < 0$. The entrepreneur therefore decides to shut down in the down state if

$$(1 - \alpha)R > \pi(1 - \alpha)s_{ud} + (1 - \pi)(s_{dd} - X), \quad \forall \pi \in \Pi, \quad (45)$$

which is equivalent to

$$(1 - \alpha)R > (1 - \alpha)E_{p+\epsilon}[\tilde{s}_2^d] - (1 - p - \epsilon)(X - \alpha s_{dd}), \quad (46)$$

or,

$$X > \alpha s_{dd} + \frac{(1 - \alpha)E_{p+\epsilon}[\tilde{s}_2^d] - R}{1 - p - \epsilon}. \quad (47)$$

From (47) we see that the first inequality in condition (40) is always satisfied. To insure that the second inequality in (40) is also satisfied, we need further restrictions on the technology parameters.

1. If $\min\{s_{dd}, \alpha R\} = s_{dd}$, i.e, $R > \frac{s_{dd}}{\alpha}$, then condition (40) requires that

$$\alpha s_{dd} + \frac{(1 - \alpha)E_{p+\epsilon}[\tilde{s}_2^d] - R}{1 - p - \epsilon} < s_{dd}, \quad (48)$$

or $R > (p + \epsilon)s_{ud}$. Hence a sufficient condition for the convertible bond to induce shutting down in the down state is that $R > \max\{\frac{s_{dd}}{\alpha}, (p + \epsilon)s_{ud}\}$. This condition states that the recovery value cannot be too low in order for the convertible bond to have effect. This is a consequence of the fact that the convertible bond contains a carrot-like contract with a negative payoff $\eta < 0$, as noted above.

2. If $\min\{s_{dd}, \alpha R\} = \alpha R$, i.e, $R < \frac{s_{dd}}{\alpha}$, then condition (40) requires that

$$\alpha s_{dd} + \frac{(1 - \alpha)E_{p+\epsilon}[\tilde{s}_2^d] - R}{1 - p - \epsilon} < \alpha R, \quad (49)$$

or

$$R > \frac{E_{p+\epsilon}[\tilde{s}_2^d] + \alpha s_{dd}}{1 - \alpha(1 - p - \epsilon)}. \quad (50)$$

Hence conditions (40) and (47) are satisfied if

$$\frac{E_{p+\epsilon}[\tilde{s}_2^d] + \alpha s_{dd}}{1 - \alpha(1 - p - \epsilon)} < R < \frac{s_{dd}}{\alpha} \quad (51)$$

Note, finally that, in the up state the incentive to expand are not affected by the presence of a convertible bond because all the payoffs are simply scaled by the factor $1 - \alpha$, as shown by (41). Therefore, under the anticipated policy in which the entrepreneur expands in state up and contract in state down, the convertible bond is a gamble b with payoffs $\lambda_2(E_{p-\epsilon}[\tilde{s}_2^u] - I_1)$ in the up state (i.e., the continuation value from expanding) and αR in the down state. It therefore follows that a CEU financier purchasing such a convertible bond will be willing to pay a price P such that

$$b - P \succ 0 \iff P < \alpha [(p - \epsilon)(\lambda_2 E_{p-\epsilon}[\tilde{s}_2^u] - I_1) + (1 - p - \epsilon)R]. \quad (52)$$

Notice that the price of the bond is independent of the face value X . The face value only plays the role of a “threat” to induce the entrepreneur to avoid taking actions that will result in having to pay X to the bondholder. The entrepreneur undertakes the investment at time $t = 0$ if

$$I_0 - P < (1 - \alpha) [(p - \epsilon)(\lambda_2 E_{p-\epsilon}[\tilde{s}_2^u] - I_1) + (1 - p - \epsilon)R]. \quad (53)$$

Using (52), inequality (53) is equivalent to

$$I_0 < (p - \epsilon)(\lambda_2 E_{p-\epsilon}[\tilde{s}_2^u] - I_1) + (1 - p - \epsilon)R, \quad (54)$$

where, again, the right hand side of (54) is exactly the payoff of the precommitment strategy P_1 defined in (30). The following proposition summarizes the above discussion and characterizes the set of convertible bonds that can be used to solve the entrepreneur’s commitment problem

Proposition 2. *Suppose Assumption 1 is satisfied and that the investment cost $I_0 > I_0^*$, where I_0^* is the threshold defined in (28). Then by issuing a convertible bond with conversion ratio $\alpha \in (0, 1)$ and face value X belonging to the set*

$$\Gamma = \{(X, s_{dd}) \times (0, 1) | \alpha s_{dd} \leq X \leq \alpha R \text{ and (47) is satisfied} \} \quad (55)$$

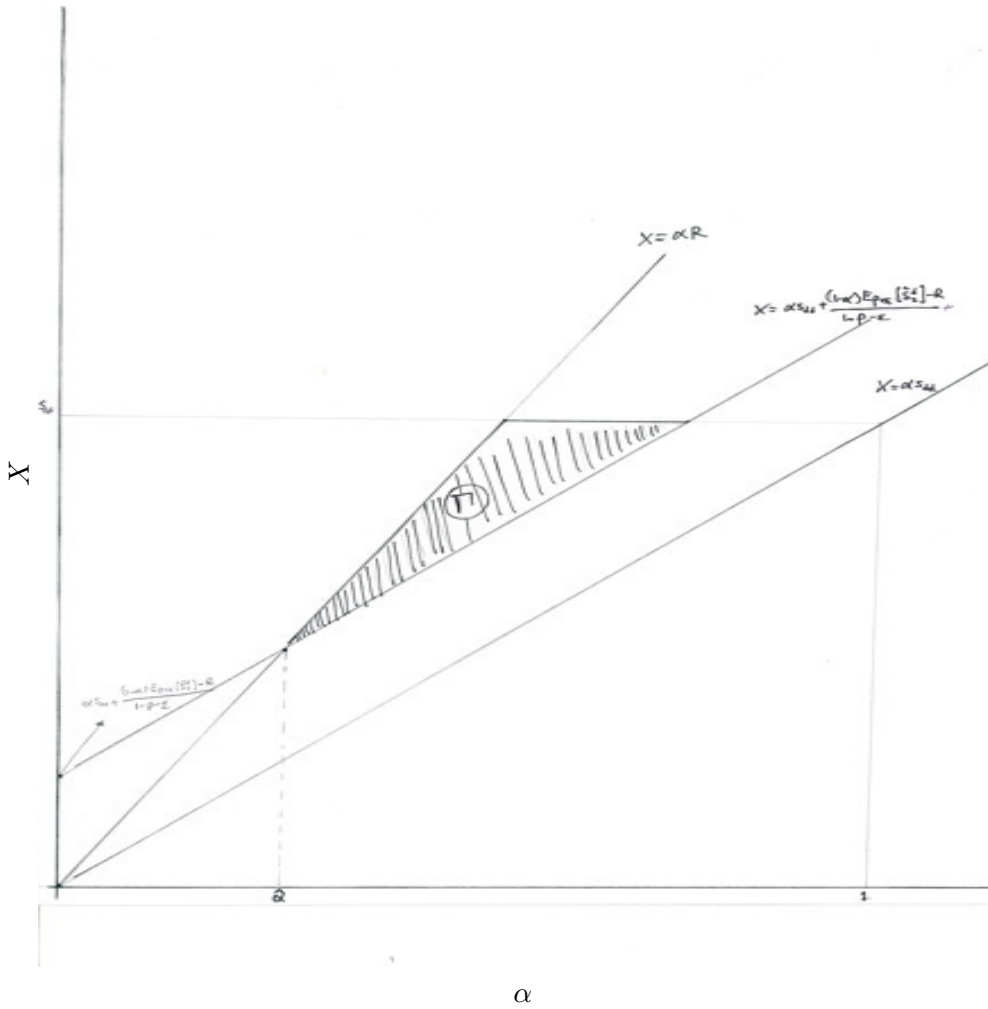
the recursive and commitment investment strategy of the entrepreneur coincide.

Figure 7 illustrates the region Γ of feasible convertible bonds. Note finally that if E need to finance the project then only a subset of Γ will satisfy the financing constraint. Specifically, the entrepreneur will have to choose a convertible bond such that $P \geq I_0$, i.e., he needs to choose a conversion ratio α such that

$$\alpha > \hat{\alpha} \equiv \frac{I_0}{[(p - \varepsilon)(\lambda_2 E_{p-\varepsilon}(\tilde{s}^u) - I_1) + (1 - p + \varepsilon)R]} \quad (56)$$

Figure 7: Convertible bond

The shaded region in the figure represents the set of convertible bond that can be used to resolve the entrepreneur's commitment problem.



4 External Financing

In this section we consider the case where the entrepreneur needs to finance the project and we start with a one period financing model. A risk neutral entrepreneur E has access to a project that delivers cash flows $\tilde{\theta}$ at $t = 1$ but has no funds and must obtain financing from a risk neutral financier F . The cash flow can take the value θ_u in the ‘up’ state and the value θ_d in the down state. We assume that, when F feels that the project is ambiguous, he thinks that the probability of the ‘up state’ (π_F) belongs to the set

$$\Pi_F = \{\pi_F : p - \varepsilon_F < \pi_F < p + \varepsilon_F\}. \quad (57)$$

Similarly, when E feels that the project is ambiguous he thinks that the probability of the ‘up state’ (π_E) belongs to the set

$$\Pi_E = \{\pi_E : p - \varepsilon_E < \pi_E < p + \varepsilon_E\}. \quad (58)$$

We consider two extreme cases. In the first case, F has multiple priors ($\varepsilon_F > 0$) and E has a single prior ($\varepsilon_E = 0$). In the second case the roles are reversed: E has multiple priors ($\varepsilon_E > 0$) and F has a single prior ($\varepsilon_F = 0$). As we mentioned, we consider the case of a single period financing arrangement in this section. The next section studies contracting in a dynamic setting.

4.1 Multi-prior F and single-prior E

Suppose that $\varepsilon_E = 0$ and $\varepsilon_F > 0$. We study the financing arrangement when F has either CEU or MEU preferences.

4.1.1 CEU preferences

When F has CEU preferences, every sale of security from E to F is a negative NPV transaction for E . In fact, under these preferences of F and the fact that E 's prior belong to Π_F , the maximum amount that F is willing to pay for a security is always smaller than the NPV of this security according to E 's prior. As a result, if E will always prefer self financing, whenever possible, to outside financing. If E does not have capital to invest, he must finance the project by issuing a security \tilde{D} delivering the cash flows D_u in the ‘up state’ and D_d in the ‘down

state'.¹⁷ Because financing is costly for E , as we show in the next proposition, he will raise just the necessary I . The set of feasible securities \mathcal{D} is

$$\mathcal{D} = \left\{ \tilde{D} \mid 0 \leq \tilde{D} \leq \tilde{\theta}, E_{\pi_F}(\tilde{D}) \geq I \text{ for all } \pi_F \in \Pi_F \right\} \quad (59)$$

E solves the following problem

$$\sup_{\tilde{D} \in \mathcal{D}} V_E(\tilde{D}) = -I + \inf_{\pi_F \in \Pi_F} E_{\pi_F}(\tilde{D}) + E_p(\tilde{\theta} - \tilde{D}). \quad (60)$$

In the above expression, $\inf_{\pi_F \in \Pi_F} E_{\pi_F}(\tilde{D})$ is the amount raised from a multi-prior CEU financier, and $E_p(\tilde{\theta} - \tilde{D})$ is E 's cash flow, net of the repayment promises from the security \tilde{D} . Because E 's belief p is an element of the set Π_F , $\inf_{\pi_F \in \Pi_F} E_{\pi_F}(\tilde{D}) < E_p(\tilde{D})$ and therefore the following inequality is always true

$$V_E(\tilde{D}) \leq -I + E_p(\tilde{\theta}), \quad (61)$$

i.e., the financing arrangement is always a negative NPV transaction. When inequality (61) binds then the financing arrangement is a zero NPV transaction. This happens in two important cases. The first case is when the set Π_F is the singleton $\Pi_F = \{p\}$ (i.e. $\varepsilon_F = 0$) and the second case is when the contract \tilde{D} has state independent payoffs, that is $D_u = D_d$. The following proposition characterizes the optimal financing arrangement.

Proposition 3. *Suppose E has single prior beliefs (SEU) and F has multiple prior beliefs on the productivity of a project and that F 's preferences are driven by CEU. If self-financing is not possible then it is optimal to finance the project with a security that $\tilde{D}^* = (D_u^*, D_d^*)$ that has the form*

$$\tilde{D}^* = (D_u^*, D_d^*) = \begin{cases} (I, I) & \text{if } 0 \leq I \leq \theta_d \\ \left(\theta_d + \frac{1}{p - \varepsilon_F}(I - \theta_d), \theta_d \right) & \text{if } \theta_d \leq I \leq E_{p - \varepsilon_F}(\tilde{\theta}) \end{cases} \quad (62)$$

If $I > E_{p - \varepsilon_F}(\tilde{\theta})$ financing is not feasible.

The intuition for the proposition is that each time E issues a security, he is going to lose money. When possible he wants to avoid issuing securities. If he must raise money, he will first

¹⁷We utilize the abuse of notation $\tilde{D} = (D_u, D_d)$ to define a security that pays the cash flow D_u in the 'up state' and D_d in the 'down state'. Furthermore, the notation \tilde{D} designates both the security and also the (random) cash flow generated by this security.

issue a security with constant payoff because their NPV is insensitive to beliefs. In fact both E and F agree on the valuation of a constant payoff security (riskless bond). If issuing riskless bonds does not allow to raise enough money to finance the investment, then E start issuing state contingent payoff securities (risky bond) up to a point where the NPV of the whole firm under the worst belief, $\pi_F = p - \varepsilon_F$, is larger than I in which case, the project is too costly and E abandons it. By issuing the security \tilde{D}^* E raises the exact amount I to invest and gets the utility

$$V^E(\tilde{D}^*) = E_p(\tilde{\theta}) - E_p(\tilde{D}^*) = \underbrace{\theta_d + p(\theta_u - \theta_d) - I}_{\text{NPV under SEU}} - \underbrace{\frac{\varepsilon_F}{p - \varepsilon_F} (I - D_d^*)}_{\text{Loss due to F's Ambiguity}}$$

The term $\frac{\varepsilon_F}{p - \varepsilon_F} (I - D_d^*)$ represents thus the additional loss that E incurs when issuing a security to an ambiguity sensitive financier. The additional loss is zero if $\varepsilon_F = 0$ or if $D_d^* = \theta_d = I$ (in which case $\tilde{D}^* = (I, I)$).

Notice that if E has some cash $W_0 < I$ then the above proposition applies by changing I to $I - W_0$. We therefore have a pecking order in the security issuance decision. The first best is to use cash to finance a project. If this is not possible, E would try to issue risk-less debt and if this also is not possible, financing will occur through risky bonds securities.

4.1.2 MEU preferences

If F has MEU preferences, then given the set of priors Π_F the maximum price that they are willing to pay for a security \tilde{D} is $\inf_{\pi_F \in \Pi_F} E_{\pi_F}(\tilde{D})$. Because

$$\inf_{\pi_F \in \Pi_F} E_{\pi}(\tilde{D}) \geq I \iff E_{\pi_F}(\tilde{D}) \geq I \text{ for all } \pi_F \in \Pi_F,$$

the feasible set of securities for the case of MEU preferences corresponds to the set (59) obtained for the case of CEU preference. Therefore Proposition 3 will also hold when F has MEU preferences and the same order for securities will prevail. We conclude then that we will observe the same type of contracts when the financiers use MEU or CEU to make decisions.

4.2 Multi-prior E and single-prior F

Suppose that $\varepsilon_E > 0$ and $\varepsilon_F = 0$. We study the financing arrangement when E has either CEU or MEU preferences. In what follows we denote by $V_E^{\pi_E}(\tilde{D})$ E 's subjective valuation of the cash flow obtained from starting the project by issuing \tilde{D} under the belief π_E .

4.2.1 CEU preferences

The set of feasible contracts is:

$$\mathcal{G} = \left\{ \tilde{D} \mid 0 \leq \tilde{D} \leq \tilde{\theta}, E_p(\tilde{D}) \geq I \right\} \quad (63)$$

Assuming E has a status quo of zero wealth, he will finance the project with the security \tilde{D} if and only if

$$V_E^{\pi_E}(\tilde{D}) = -I + E_p(\tilde{D}) + E_{\pi_E}(\tilde{\theta} - \tilde{D}) > 0 \quad \text{for all } \pi \in \Pi_F. \quad (64)$$

This constraint defines the set of implementable securities. By (63), for every feasible contract $\tilde{D} \in \mathcal{G}$, we have $\tilde{\theta} \geq \tilde{D}$, and therefore any feasible security $\tilde{D} \in \mathcal{G}$ satisfies the constraint (64) and can be implemented. The only restrictive constraint is the financing constraint $E_p(\tilde{D}) \geq I$. If this constraint is satisfied, then E is willing to start the project with any financing arrangement $\tilde{D} \in \mathcal{G}$. The following proposition summarizes the financing arrangement in this case.

Proposition 4. *Suppose F has single prior beliefs (SEU) and E has multiple prior beliefs and CEU preferences. If $I \leq E_p(\tilde{\theta})$, the entrepreneur chooses to start the project by issuing any security $\tilde{D} \in \mathcal{G}$. Due to incomplete preferences, the entrepreneur is unable to rank the different financing options.*

The set \mathcal{G} of feasible contracts is described in Figure 8. Notice that this set always contains the point $\tilde{D} = (\theta_u, \theta_d)$ which corresponds to the equity contract that sells the entire project to F . Note also that for $\theta_d < I$ (Panel A), riskless debt $\tilde{D} = (I, I)$ is not feasible while for $\theta_d > I$ (Panel B), riskless debt is one of the possible financing arrangements.

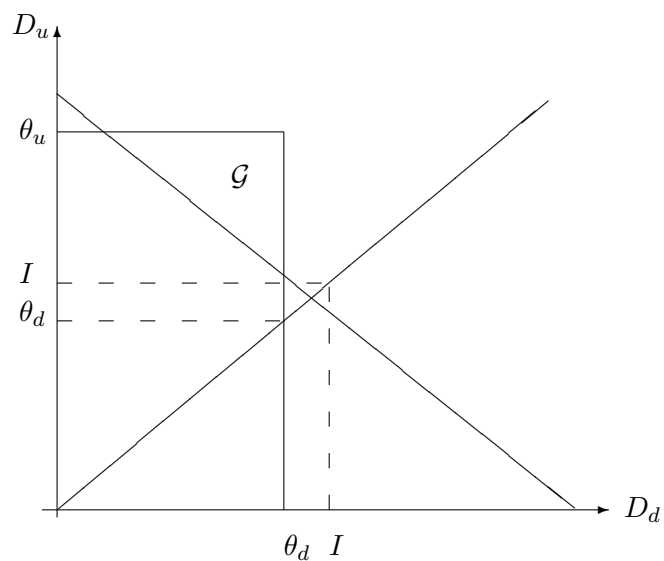
Collecting the result from Proposition 3 and Proposition 4, we see that it is the financier's set of priors which determine whether the project is started.¹⁸ Notice also that the optimal security

¹⁸This is because the entrepreneur's has no attractive outside options in our model and as a result, if financing is possible E will be better off starting the project. If we make the alternative assumption that E must give up some opportunities if he start the project, then his preference will have a more important impact on the decision to start the project.

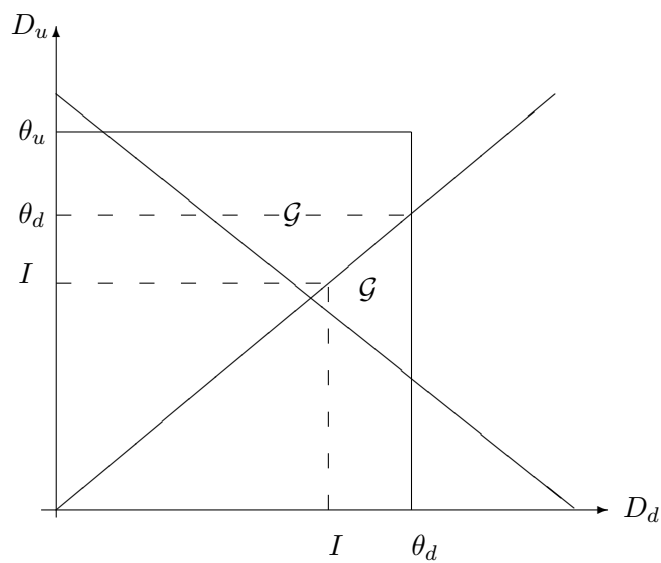
Figure 8: Feasible contracts, CEU case.

The figure displays the set \mathcal{G} of feasible contracts for the case in which E exhibit CEU preferences and F is ambiguity neutral.

Panel A: $\theta_d < I$



Panel B: $\theta_d > I$



from Proposition 3 is contained the set \mathcal{G} is all subcases. Thus financing with this contract is an acceptable option for E for all configuration of preferences for E and F.

4.2.2 MEU Preferences

When E makes MEU choices and F makes SEU choices, the set of feasible contracts is:

$$\mathcal{G} = \left\{ \tilde{D} \mid 0 \leq \tilde{D} \leq \tilde{\theta}, E_p(\tilde{D}) \geq I \right\}$$

E chooses the optimal contract by solving the following optimization problem

$$\sup_{\tilde{D} \in \mathcal{G}} V_E(\tilde{D}) = -I + E_p(\tilde{D}) + \inf_{\pi_E \in \Pi_E} E_{\pi_e}(\tilde{\theta} - \tilde{D}),$$

E will start the project whenever this quantity is positive.

Proposition 5. *When E has multiple prior beliefs and MEU preferences and F has single prior beliefs, E will start the project if and only if*

$$I \leq E_p(\tilde{\theta}).$$

If this condition holds, then it is optimal to sell the whole firm, i.e., $\tilde{D} = (\theta_u, \theta_d)$.

There are multiple optima because Figure 8 shows that the set \mathcal{G} intersects the set of contracts leaving E with flat payoff ($\theta_u - D_u = \theta_d - D_d$) then any element D of this intersection gives E the value

$$V_E(\tilde{D}) = -I + E_p(\tilde{D}) + \inf_{\pi} E_{\pi}(\tilde{\theta} - \tilde{D}) = -I + E_p(\tilde{D}) + E_p(\tilde{\theta} - \tilde{D}) = -I + E_p(\tilde{\theta})$$

and therefore the security \tilde{D} is also an optimal choice.

4.3 Multi-prior F and E

In this section we assume that E has a multiple prior set Π_E and F has a multiple prior set Π_F and we make the assumption that

$$\Pi_E \subseteq \Pi_F \iff \varepsilon_E \leq \varepsilon_F.$$

4.3.1 CEU preferences

We assume that both F and E have CEU preferences. The set of contracts satisfying the financing constraints is

$$\mathcal{H} = \left\{ \tilde{D} \mid 0 \leq \tilde{D} \leq \tilde{\theta} \text{ and } \inf_{\pi_F \in \Pi_F} E_{\pi}(\tilde{D}) \geq I \right\}$$

If E finances the project with $\tilde{D} \in \mathcal{H}$, his subjective valuation under prior $\pi_E \in \Pi_E$ is

$$V_E^{\pi_E}(\tilde{D}) = -I + \inf_{\pi_F \in \Pi_F} E_{\pi_F} \tilde{D} + E_{\pi_E}(\tilde{\theta} - \tilde{D}).$$

Because $\tilde{D} \leq \tilde{\theta}$ any contract from \mathcal{H} is implementable and E is willing to start the firm and finance it with any $\tilde{D} \in \mathcal{H}$. In the next proposition we show that the structure of the set Π_E is irrelevant for this result provided that $\Pi_E \subseteq \Pi_F$. Now unlike the case where F has a single prior beliefs, it is possible to rank the financing contracts as the following proposition shows.

Proposition 6. *The optimal financing arrangement when both E and F have the multiple priors $\Pi_E \subset \Pi_F$ and have CEU preferences is as follow.*

1. *If $I > E_{p-\varepsilon_F}(\tilde{\theta})$, then E cannot finance the project.*
2. *If $\theta_d \leq I \leq E_{p-\varepsilon_F}(\tilde{\theta})$ then E will accept to finance the project with any \tilde{D} in the set \mathcal{H} .¹⁹ Moreover, all the contracts in the set \mathcal{H} are dominated (from E 's perspective) by the contract*

$$\tilde{D}^* = \left(\theta_d + \frac{1}{p-\varepsilon} (I - \theta_d), \theta_d \right).$$

3. *If $0 \leq I \leq \theta_d$, then E will accept to finance the project with any \tilde{D} in the set \mathcal{H} . Moreover, all the contracts in the set \mathcal{H} are dominated (from E 's perspective) by the risk free contracts of the form*

$$\tilde{D} = (\gamma, \gamma) \quad \text{with } I \leq \gamma \leq \theta_d.$$

¹⁹Notice that when $\theta_d \leq I \leq E_{p-\varepsilon}$ the set \mathcal{H} can also be defined as

$$\mathcal{H} = \{D \mid D \leq \theta \text{ and } E_{p-\varepsilon}(D) \geq I\}$$

Notice that in all subcases, the only dominating contracts are the one which are optimal when E uses SEU to make decisions and F uses CEU to make decisions. The above proposition says that if we only select the dominating contracts, the set of priors of E is irrelevant in the context of our problem provided that it is included in the F 's set of priors.

Notice that the assumption $\Pi_E \subseteq \Pi_F$ seems crucial for the above proposition. Under this assumption the structure of the set of E 's prior is irrelevant and everything is as if E has a single prior and SEU preferences. It can be shown that if we assume instead that $\Pi_F \subseteq \Pi_E$ we will have multiple contract which are not comparable as in the case where F has a single prior (section 4.2).

4.3.2 MEU preferences

Here again, the assumptions on the sets Π_E and Π_F are going to be important. If the set Π_F is the largest then E will issue risk free securities when they satisfy the financing constraints. If this is not possible they will issue a state contingent security that gives up all the firm in the down state. If the set Π_E is the largest then, E will sell the whole firm. TBC

5 Dynamic contracting under ambiguity

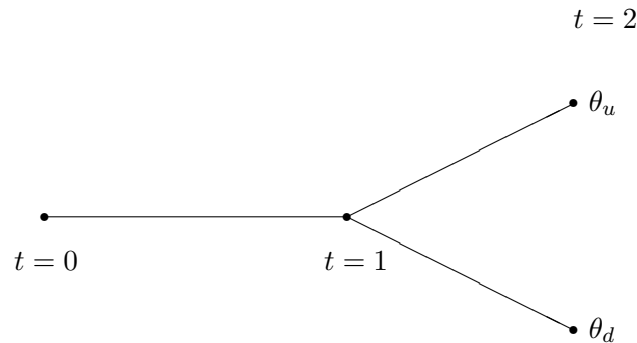
In Section ?? we showed how the change is status quo together with the multi-prior feature of CEU can generate time inconsistency in E 's decisions. This time inconsistency emerges endogenously with CEU in a dynamic context and creates a new economic force that may generate a demand for any security that will help E to commit to the policy that he currently prefers. The demand for this particular form of securities does not exist under SEU or MEU preferences. The purpose of this section is to illustrate how the demand for this type of securities can be addressed in an optimal contract setting.

E has access to an investment opportunity generating a cash flow θ_u in the “up” state and θ_d in the “down” state at time $t = 2$. The cash flow of the project is then $\theta = (\theta_u, \theta_d)$. A cost of I needs to be paid to begin the project and E does not have enough cash to pay for it. To simplify the model, we assume that the E has no personal wealth and if he wants to start the project, he must issue some securities to the financiers (F) in order to start the project. The securities are issued at $t = 0$ and they have an option feature that can be exercised at the intermediate

time $t = 1$. There is no new information being revealed between time $t = 0$ and $t = 1$, i.e. the information structure is summarized by Figure 11.

Figure 9: Information structure for dynamic contracting

The figure displays the information structure we consider in studying the dynamic contracting problem in the presence of ambiguity. Securities are issued at time $t = 0$ and they contain an option feature that can be exercised at time $t = 1$. Payoffs are received at time $t = 2$. No information is revealed between time $t = 0$ and $t = 1$.



We suppose that F has multiple priors in the set $\Pi = [p - \varepsilon, p + \varepsilon]$ and we will consider both MEU and CEU preferences. At this stage, we do not commit to any preferences for E (we just discuss valuation and not contracting).

We will consider two type of primitive contracts. Risky debt has the form

$$B^\beta = (B_u^\beta = \theta_d + \beta(\theta_u - \theta_d), B_d^\beta = \theta_d) \text{ with } 0 \leq \beta \leq 1.$$

When $\beta = 0$, the debt is safe and when $\beta > 0$ the debt is risky and has a face value B_u^β .

Equity has the form

$$Q^\alpha = (\alpha\theta_u, \alpha\theta_d) \text{ with } 0 \leq \alpha \leq 1.$$

Because the payoff in the up state is always larger for the class of securities that we consider, we know that if F makes decisions based on CEU or MEU, E will be able to raise the amount

$$E_{p-\varepsilon}(D), \text{ for } D = B^\beta \text{ or } D = Q^\alpha.$$

Notice that the amount raised is equal under CEU or MEU.

Now we turn to the cases where E issues a security with an option feature. Consider the security $BQ^{\alpha,\beta}$ which gives to the savers the possibility to convert the bond B^α to the equity Q^α . The conversion decision must be taken at time $t = 1$. Similarly, we consider the security $QB^{\alpha,\beta}$ which gives the savers to convert the equity to a bond at time $t = 1$. With both securities, F must decide if they exercise the option at time $t = 1$ by comparing the value under a particular prior π of the bond

$$\pi(\theta_d + \beta(\theta_u - \theta_d)) + (1 - \pi)(\theta_d) = \theta_d + \pi\beta(\theta_u - \theta_d)$$

with the value of the equity

$$\alpha\pi\theta_u + \alpha(1 - \pi)\theta_d.$$

With MEU, the decision is based on the worst prior $p - \varepsilon$ and the indifference frontier in the plan (α, β) is given by

$$\alpha = \frac{\theta_d + (p - \varepsilon)\beta(\theta_u - \theta_d)}{E_{p-\varepsilon}(\theta)}$$

The frontier splits the plan into a region Γ_1 where debt is preferred and a region Γ_2 where equity is preferred.

Therefore, both securities $BQ^{\alpha,\beta}$ and $QB^{\alpha,\beta}$ will have the same price which is given by

$$\text{Proceeds}(BQ^{\alpha,\beta}) = \text{Proceeds}(QB^{\alpha,\beta}) = \theta_d + (p - \varepsilon)\beta(\theta_u - \theta_d) \text{ if } (\alpha, \beta) \in \Gamma_1$$

and

$$\text{Proceeds}(BQ^{\alpha,\beta}) = \text{Proceeds}(QB^{\alpha,\beta}) = \alpha E_{p-\varepsilon}(\theta) \text{ if } (\alpha, \beta) \in \Gamma_2$$

With CEU, the plan is split into three regions Γ'_1 , Γ''_1 and Γ_2 .

In the region Γ''_1 debt is preferred and in region Γ_2 equity is preferred. Region Γ'_1 is a region where equity and debt are not comparable (under some prior debt is preferred whereas under some other priors equity is preferred). The region Γ'_1 is described by the equation

$$\frac{\theta_d + (p + \varepsilon)\beta(\theta_u - \theta_d)}{E_{p+\varepsilon}(\theta)} \leq \alpha \leq \frac{\theta_d + (p - \varepsilon)\beta(\theta_u - \theta_d)}{E_{p-\varepsilon}(\theta)}$$

In the regions Γ''_1 and Γ_2 the proceeds are given as before

Figure 10: Conversion regions, MEU case.

The figure displays the conversion regions in the case of MEU preferences. In region Γ_1 debt is preferred and in region Γ_2 equity is preferred.

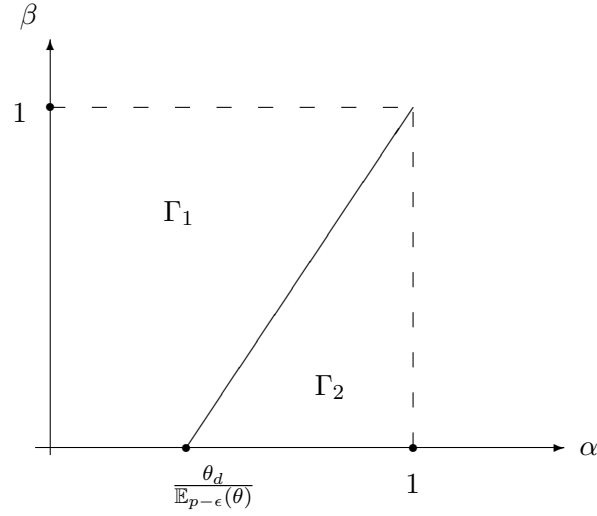
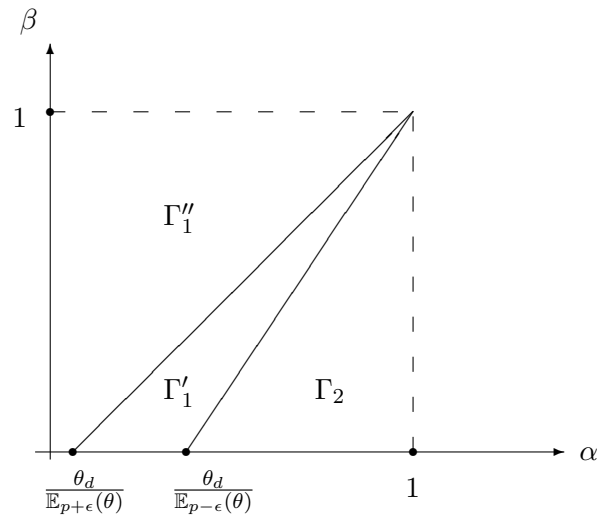


Figure 11: Conversion regions, CEU case.

The figure displays the conversion regions in the case of CEU preferences. In region Γ''_1 debt is preferred, in region Γ_2 equity is preferred and in region Γ'_1 equity and debt are not comparable.



$$\text{Proceeds}(BQ^{\alpha,\beta}) = \text{Proceeds}(QB^{\alpha,\beta}) = \begin{cases} \theta_d + (p - \varepsilon)\beta(\theta_u - \theta_d) & \text{if } (\alpha, \beta) \in \Gamma''_1 \\ \alpha E_{p-\varepsilon}(\theta) & \text{if } (\alpha, \beta) \in \Gamma_2 \end{cases}$$

But on the region Γ'_1 , the amount raised by E may change with the security that is issued. When pricing Security $BQ^{\alpha,\beta}$ with $(\alpha, \beta) \in \Gamma'_1$ at time $t = 0$, F knows that he will not exercise the conversion option because he will have a bond as a status quo. As a result, he will price this security as a straight bond. More formally, for any $(\alpha, \beta) \in \Gamma'_1$

$$\text{Proceeds}(BQ^{\alpha,\beta}) = \theta_d + (p - \varepsilon)\beta(\theta_u - \theta_d).$$

Similarly, F will price the security $QB^{\alpha,\beta}$ as an equity and for any $(\alpha, \beta) \in \Gamma'_1$,

$$\text{Proceeds}(QB^{\alpha,\beta}) = \alpha E_{p-\varepsilon}(\theta).$$

This result suggests that when issuing convertible equities (an equity with the option to convert it to a bond), it is possible E will be able to raise more money when F make decisions based on MEU than when they make decision based on CEU.

Proposition 7. *For any $(\alpha, \beta) \in \Gamma'_1$, the proceeds from selling $QB^{\alpha,\beta}$ are larger when savers are MEU than when they are CEU. More formally,*

$$\text{Proceeds}^{CEU}(QB^{\alpha,\beta}) = \alpha E_{p-\varepsilon}(\theta) \leq \text{Proceeds}^{MEU}(QB^{\alpha,\beta}) = \theta_d + (p - \varepsilon)\beta(\theta_u - \theta_d)$$

The result in this proposition is a manifestation of the asymmetry that we already observed in the expansion/contraction options example of section ???. The proposition says that some securities with option features will be overvalued by MEU savers relative to CEU savers.

Proposition 7 illustrates an important difference between CEU and MEU. Consider a hybrid security A offering the ownership of security X with the option to convert it one period to security Y . Assume securities X and Y pay the cash flows after the first period. Consider the alternative hybrid security B offering the ownership of security Y with the option to convert it one period to security X . Assume that there is no information revelation about cash flows between the issuance date and the option exercise date.

When F uses SEU with the prior $\pi = p$, the *ex ante* valuation of the two securities is clearly identical and is given by

$$\text{Proceeds}(A) = \text{Proceeds}(B) = \text{Max} \{E_p(X), E_p(Y)\}$$

Notice that the security valuation at $t = 0$ is also identical if it was possible to commit to a particular policy exercise. From the perspective of time $t = 0$, the SEU saver will also pick the security delivering the highest expected payoff under the probability p even when commitment is possible.

When F is MEU, the *ex ante* valuation of the two securities is also identical and is given by

$$\text{Proceeds}(A) = \text{Proceeds}(B) = \text{Max} \left\{ \inf_{\pi} E_{\pi}(X), \inf_{\pi} E_{\pi}(Y) \right\}$$

The security valuation at time $t = 0$ does not change if it was possible for saver to commit to a particular policy decision. In this case, the MEU saver will still pick the security that offers the highest expected payoff according to the worst probability measure.

With CEU the valuation of security A can be different from the valuation of Security B. Specifically, when the primitive security X is not comparable with security Y , the inertia assumption shows that

$$\text{Proceeds}(A) = \inf_{\pi} E_{\pi}(X)$$

whereas

$$\text{Proceeds}(B) = \inf_{\pi} E_{\pi}(Y)$$

This is a first important difference between MEU and CEU: it seems that CEU is sensitive to the sequencing of the options of hybrid securities whereas sequencing is irrelevant for SEU and MEU. The sequence of option is relevant for CEU because it induces a particular path of status quo which in turn break down the indifference in the exercise choice.

The second difference is that when commitment on the exercise policy is possible, the valuation of the security A can be different from the valuation of the security A in the absence of commitment. An interesting situation occurs when X is not comparable to 0 whereas X dominates 0.²⁰ In this case, the commitment exercise policy for the hybrid security A is to convert it to Y because the commitment solution uses 0 as a status quo. As a result, the valuation of security A under commitment is given by

$$\text{Proceeds}(A) = \inf_{\pi} E_{\pi}(Y)$$

²⁰This can occur when $X = (-1, 3)$ and $Y = (1, 2)$. With a large enough set of priors, we see that X is not comparable with 0. The security Y always dominates 0. Again with a large enough set of priors X is not comparable with Y .

and it is different from its valuation when commitment is not possible.

6 Conclusion

We have examined the way in which ambiguity aversion, modeled as multiple priors, affects real investment and financing decisions. We considered two approaches to decision making under ambiguity which derive from two different relaxation of Savage's Subjective expected utility paradigm: the MEU approach, based on the relaxation of the independence axiom, and the CEU, based on the relaxation of the completeness axiom. The two approaches can deliver considerably different, and sometime contradictory investment and financing decisions. Such heterogeneity in predicted choices can be useful for future empirical work that would attempt to determine which approach better describe observed corporate behavior over time.

A Appendix. Decision theory toolkit

In this appendix we review the theoretical foundations for the three decision models used in the paper, Subjective Expected Utility (SEU), Minimum Expected Utility (MEU) and Consensus Expected Utility (CEU). Our analysis is cast in the framework developed by Anscombe and Aumann (1963) and draws heavily on the review article by Gilboa and Marinacci (2011).

A.1 Preliminaries

Let S denote a set describing the possible states of the world, endowed with an event algebra Σ , X be a set describing possible outcomes, and $\Delta(X)$ the space of *lotteries* over outcomes X . A lottery is a random variable with outcomes in X whose probabilities are *objectively* known by the decision maker (DM). For simplicity of exposition, we take S and X as discrete sets.

The DM makes choices over *acts*, which are functions that map states into lotteries. Formally, an act f is a Σ -measurable function $f : S \rightarrow \Delta(X)$. Because the space $\Delta(X)$ is convex one can construct convex combination of acts, i.e., given any two acts f and g and $\alpha \in [0, 1]$ the mixed act $\alpha f + (1 - \alpha)g$ is a function from S to $\Delta(X)$ defined as

$$(\alpha f + (1 - \alpha)g)(s) = \alpha f(s) + (1 - \alpha)g(s), \quad \forall s \in S. \quad (\text{A1})$$

Let \mathcal{F} be the space of acts. The DM has preferences \succeq on \mathcal{F} , i.e., $f \succeq g$ means that act f is weakly preferred to g . Note that a preference \succeq over acts induces a preference \succeq_{Δ} over lotteries if, for all p and q in $\Delta(X)$ one defines

$$p \succeq_{\Delta} q \iff f \succeq g, \quad (\text{A2})$$

where f and g are constant acts, i.e., $f(s) = p$ and $g(s) = q$ for all $s \in S$. Because constant acts are not subject to state uncertainty, the preference \succeq_{Δ} captures preference with respect to *risk*, as opposed to uncertainty.

Example 2. Let us consider a risk neutral manager who has I dollars and considers whether to invest in a project that yield random outcomes or not. Suppose there are two possible state of nature ‘up’ and ‘down’. In state ‘up’ the project outcome is x_u and in state ‘down’ the outcome is x_d . If the manager does not invest, the outcome is simply I .

In terms of the above notation, the state space is $S = \{down, up\}$ and the outcome space is $X = \{x_u, x_d, I\}$. “Investing” is an act f such that:

$$f(u) = \begin{cases} x_u, & \text{with prob. 1} \\ x_d, & \text{with prob. 0} \\ I, & \text{with prob. 0} \end{cases}, \quad f(d) = \begin{cases} x_u, & \text{with prob. 0} \\ x_d, & \text{with prob. 1} \\ I, & \text{with prob. 0} \end{cases} \quad (\text{A3})$$

while “Not investing” is an act g such that

$$g(u) = \begin{cases} x_u, & \text{with prob. 0} \\ x_d, & \text{with prob. 0} \\ I, & \text{with prob. 1} \end{cases}, \quad g(d) = \begin{cases} x_u, & \text{with prob. 0} \\ x_d, & \text{with prob. 0} \\ I, & \text{with prob. 1} \end{cases}. \quad (\text{A4})$$

In other words f and g are degenerate lotteries, which is a formal way to express the fact that upon investing we know *for sure* the outcome we obtain in each state, although we do not know a priori which one of the two states, ‘up’ or ‘down’ will materialize. ■

In the sequel we summarize the axioms on the primitive preferences \succeq on which the three model of choice we consider in the paper rest, and provide the representation of the preference relation \succ .

A.2 Subjective Expected Utility (SEU)

Savage’s SEU model of decision making rests on the following axioms:

S1. **Weak order.** \succeq on \mathcal{F} is complete and transitive.

S2. **Monotonicity.** For any $f, g \in \mathcal{F}$, if $f(s) \succeq_{\Delta} g(s)$ for each $s \in S$, then $f \succeq g$.

S3. **Independence.** For any $f, g, h \in \mathcal{F}$, and any $\alpha \in (0, 1)$ we have

$$f \succ g \Rightarrow \alpha f + (1 - \alpha)h \succ \alpha g + (1 - \alpha)h \quad (\text{A5})$$

S4. **Archimedean.** Let $f, g, h \in \mathcal{F}$ be such that $f \succ g \succ h$. Then there are $\alpha, \beta \in (0, 1)$ such that

$$\alpha f + (1 - \alpha)h \succ g \succ \beta f + (1 - \alpha)h \quad (\text{A6})$$

S5. **Nondegeneracy.** There are $f, g \in \mathcal{F}$ such that $f \succ g$.

The following theorem characterize the SEU representation of preferences.

Theorem 3. *Let \succeq be a preference defined on \mathcal{F} . The following conditions are equivalent:*

1. *The preference \succeq satisfies the axioms S1–S5.*
2. *There exist a non constant function $u : X \rightarrow \mathbb{R}$ and a unique probability measure $\pi : \Sigma \rightarrow [0, 1]$ such that for all $f, g \in \mathcal{F}$,*

$$f \succeq g \iff \sum_{s \in S} \left(\sum_{x \in \text{supp}f(s)} u(x)f(s) \right) \pi(s) \geq \sum_{s \in S} \left(\sum_{x \in \text{supp}g(s)} u(x)g(s) \right) \pi(s) \quad (\text{A7})$$

In the case of the risk neutral manager considered in Example 2, the two acts: invest, f and don't invest, g , map into degenerate lotteries on the outcome space and hence the above theorem implies that

$$\text{Invest} \succeq \text{Don't invest} \iff \mathbb{E}^\pi[\tilde{x}] \geq I, \quad (\text{A8})$$

where $\mathbb{E}^\pi[u(\tilde{x})] = \pi(u)x_u + \pi(d)x_d$.

A.3 Minimum Expected Utility (MEU)

Gilboa and Schmeidler (1989) start from Savage axioms and:

- i. Replace the independence axiom S3 over acts with the following independence axiom over lotteries

M3. **C-Independence.** For all acts $f, g \in \mathcal{F}$ and all constant acts (lotteries) p

$$f \succ g \Rightarrow \alpha f + (1 - \alpha)p \succ \alpha g + (1 - \alpha)p, \quad \forall \alpha \in [0, 1] \quad (\text{A9})$$

- ii. Introduce a new axiom to capture aversion to uncertainty

M6. **Uncertainty Aversion.** For any $f, g \in \mathcal{F}$ and any $\alpha \in (0, 1)$

$$f \sim g \Rightarrow \alpha f + (1 - \alpha)g \geq f. \quad (\text{A10})$$

The following theorem characterize the MEU representation of preferences.

Theorem 4. *Let \succeq be a preference defined on \mathcal{F} . The following conditions are equivalent:*

1. *The preference \succeq satisfies the axioms S1, S2, M3, S4, S5, M6.*
2. *There exist a non constant function $u : X \rightarrow \mathbb{R}$ and a convex and compact set $\Pi \subseteq \Delta(\Sigma)$ of probability measures such that for all $f, g \in \mathcal{F}$,*

$$f \succeq g \iff \min_{\pi \in \Pi} \sum_{s \in S} \left(\sum_{x \in \text{supp}f(s)} u(x)f(s) \right) \pi(s) \geq \min_{\pi \in \Pi} \sum_{s \in S} \left(\sum_{x \in \text{supp}g(s)} u(x)g(s) \right) \pi(s) \quad (\text{A11})$$

In the case of the risk neutral manager considered in Example 2 the above theorem implies that

$$\text{Invest} \succeq \text{Don't invest} \iff \min_{\pi \in \Pi} \mathbb{E}^\pi[\tilde{x}] \geq I, \quad (\text{A12})$$

where $\mathbb{E}^\pi[u(\tilde{x})] = \pi(u)x_u + \pi(d)x_d$.

A.4 Consensus Expected Utility (CEU)

Bewley (2002) start from Savage's axioms and replaces the completeness axiom over act with a completeness axiom over lotteries. Formally,

C1. **C-Completeness.** For every constant act (lottery) $p, q \in \Delta(X)$, $p \succeq q$ or $q \succeq p$.

The following theorem characterize the CEU representation of preferences.²¹

Theorem 5. *Let \succeq be a preference defined on \mathcal{F} . The following conditions are equivalent:*

1. *The preference \succeq satisfies the axioms C1 and S2–S5.*
2. *There exist a non constant function $u : X \rightarrow \mathbb{R}$ and a convex and compact set $\Pi \subseteq \Delta(\Sigma)$ of probability measures such that for all $f, g \in \mathcal{F}$,*

$$f \succeq g \iff \sum_{s \in S} \left(\sum_{x \in \text{supp}f(s)} u(x)f(s) \right) \pi(s) \geq \sum_{s \in S} \left(\sum_{x \in \text{supp}g(s)} u(x)g(s) \right) \pi(s), \quad \forall \pi \in \Pi. \quad (\text{A13})$$

²¹The theorem is based on the characterization provided by Gilboa, Maccheroni, Marinacci, and Schmeidler (2010) who represent weak preferences. Bewley (2002) provides characterization of strict preferences.

In the case of the risk neutral manager considered in Example 2 the above theorem implies that

$$\text{Invest} \succeq \text{Don't invest} \iff \mathbb{E}^\pi[u(\tilde{x})] \geq I, \quad \forall \pi \in \Pi, \quad (\text{A14})$$

where $\mathbb{E}^\pi[\tilde{x}] = \pi(u)x_u + \pi(d)x_d$.

B Appendix. Proofs

Proposition 8. *Let $\Pi_0 = [\underline{\pi}_0, \bar{\pi}_0]$, $\Pi_1^u = [\underline{\pi}_1^u, \bar{\pi}_1^u]$, and $\Pi_1^d = [\underline{\pi}_1^d, \bar{\pi}_1^d]$. Then given any gamble f with outcomes f_{uu}, f_{ud}, f_{du} and f_{dd} , the set of probabilities $(\pi_0, \pi_1^u, \pi_1^d) \in \Pi_0 \times \Pi_1^u \times \Pi_1^d$ that minimizes the expectation $E_{\pi_0, \pi_1^u, \pi_1^d}[f]$, always contains one of the corners of the set $\Pi_0 \times \Pi_1^u \times \Pi_1^d$.*

Proof: ■

Proof of Proposition ??

To describe the recursive solution, we just need to specialize the analyzes in sections is readily available from ??. At time $t = 1$, the contraction decision is characterized by the inequality (?). Taking the investment policy at time $t = 1$ as a constraint, the recursive investment policy at time $t = 0$ is to invest if and only if

$$I_0 < (p - \varepsilon)\bar{\theta} + (1 - p + \varepsilon)S$$

when $S > E_{p+\varepsilon}(\tilde{\theta}^d)$ and to invest if and only if

$$I_0 < (p - \varepsilon)\bar{\theta} + (1 - p + \varepsilon)E_{p-\varepsilon}(\tilde{\theta}^d).$$

when $S \leq E_{p+\varepsilon}(\tilde{\theta}^d)$.

Under the commitment investment policy E must make one decision at time $t = 0$. E chooses between three policies: not investing (P0), Investing and continuing P1 and, investing and contracting P2. E takes P1 vs P0 if and only if

$$I_0 < (p - \varepsilon)\bar{\theta} + (1 - p + \varepsilon)E_{p-\varepsilon}(\tilde{\theta}^d) \equiv E_{p-\varepsilon}(\tilde{C}_2).$$

E takes P2 versus P0 if and only if

$$I_0 < (p - \varepsilon)\bar{\theta} + (1 - p + \varepsilon)S$$

It is easy to see with a figure that in Region \mathcal{R}_0 E wants to invest and contract (P2) and rejects the investment-continuation policy P1. In the sub-region $\mathcal{R}_0 \cap \mathcal{R}_1$, we see that the commitment investment policy requires to invest at time $t = 0$ and contract at time $t = 1$.

However, in the region \mathcal{R}_1 the recursive investment policy requires to not invest at all at $t = 0$. In fact, with the recursive approach E knows that if he invests at time $t = 0$, he will not contract at time $t = 1$: Under the condition $S < E_{p+\varepsilon}(\tilde{\theta}^d)$ there are always some optimistic beliefs that make the continue option more attractive than the contraction option. As a result, he only considers the path where he does not contract and this path is dominated by not investing in presence of the pessimistic beliefs. On the other hand, under commitment, E has no concern for the comparison between S and θ^d because he can commit to a contraction. In the region \mathcal{R}_1 , the salvage value S is large enough (resp. I_0 is large enough) to make sure that the contraction option is acceptable (resp the continue option is rejected). ■

Proof of Proposition 3

We start with the observation that an optimal security must bind the constraint

$$\inf_{\pi \in \Pi_F} E_{\pi}(\tilde{D}) = I$$

If $\inf_{\pi \in \Pi_F} E_{\pi}(\tilde{D}) > I$, then we can decrease the payoff of \tilde{D} by a small amount in one of the two states and we will then improve E 's utility.²²

Now, if $I \leq \theta_d$, then $\tilde{D} = (I, I) \in \mathcal{D}$ and E 's utility is maximized because Inequality (61) binds. On the other hand, if $I > E_{p-\varepsilon}(\tilde{\theta})$, then E is not going to be able to finance the project even if he sells the entire firm.

²²To clarify this point, suppose that $D_u > D_d$, then the inf of $E_{\pi}(\tilde{D})$ is attained at $\pi = p - \varepsilon$. Consider the security $\tilde{D}' = (D_u - \eta, D_d)$ with η being a small number so that $E_{p-\varepsilon}(\tilde{D}') \geq I$. It can be checked that E 's utility derived from issuing \tilde{D}' is given by

$$V_E(\tilde{D}') = V_E(\tilde{D}) + \varepsilon\eta > V_E(\tilde{D}).$$

and therefore the entrepreneur prefers to issue \tilde{D}' . A similar reasoning can be used when $D_u > D_d$ by considering the security $\tilde{D}'' = (D_u, D_d - \eta)$.

If $\theta_d \leq I \leq E_{p-\varepsilon_F}(\tilde{\theta})$ then it possible to finance the project. Assume that E finance with the project with a security that has $D_d < \theta_d$. It is then necessary to have $D_u > \theta_d$ to be able to raise I (because $I \geq \theta_d$). Because $D_u > D_d$, we have $I = \inf_{\pi \in \Pi_F} E_\pi(\tilde{D}) = E_{p-\varepsilon_F}(\tilde{D})$. We will now show that a small modification of \tilde{D} allows E to increase the project valuation. Consider the security \tilde{D}' defined by

$$D'_d = D_d + \eta, \quad D'_u = D_u - \eta \frac{p - \varepsilon_F}{1 - p + \varepsilon_F}$$

where η is a very small number. By construction, issuing \tilde{D}' allows to raise exactly I because $\inf_{\pi \in \Pi_F} E_\pi(\tilde{D}') = E_{p-\varepsilon_F}(\tilde{D}) = I$. It is also easy to verify that,

$$V_E(\tilde{D}') = V_E(\tilde{D}) + \eta \frac{\varepsilon_F}{1 - p + \varepsilon_F}$$

and thus $V_E(\tilde{D}') > V_E(\tilde{D})$. We conclude then that the optimal security satisfies $D_d = \theta_d$ and the financing constraint requires that

$$D_u = \theta_d + \frac{1}{p - \varepsilon_F} (I - \theta_d).$$

■

Proof of Proposition 4

We have already observed that any security D in the set \mathcal{G} is going to generate a positive $V_E^\pi(\tilde{D})$ under any π in the set of E 's priors Π_E and thus the entrepreneur is happy to start the project by issuing \tilde{D} .

To show that these contracts are not comparable, let us consider a security \tilde{D} in \mathcal{G} . We have $V_E^\pi(\tilde{D}) = -I + E_p(\tilde{D}) + E_\pi(\tilde{\theta} - \tilde{D})$ for any $\pi \in \Pi_E$. Define the security $\tilde{D}' = (D_u + \eta, D_d)$ with η small enough so that $\tilde{D}' \in \mathcal{G}$. Direct calculations show $V_E^\pi(\tilde{D}') = -I + E_p(\tilde{D}) + E_\pi(\tilde{\theta} - \tilde{D}) + \eta(p - \pi)$ and thus

$$V_E^\pi(\tilde{D}') - V_E^\pi(\tilde{D}) = \eta(p - \pi)$$

which can be positive for some π and negative for some other π provided that p is the interior of Π_E . We thus conclude that \tilde{D} and \tilde{D}' are not comparable for E . ■

Proof of Proposition 5

If E finance the firm by issuing $\tilde{D} \in \mathcal{G}$, then he derives the utility

$$V_E(\tilde{D}) = -I + E_p(\tilde{D}) + \inf_{\pi \in \Pi_E} E_\pi(\tilde{\theta} - \tilde{D}).$$

If he issues $\tilde{D} = \tilde{\theta}$, then the derived utility is

$$V_E(\tilde{\theta}) = -I + E_p(\tilde{\theta}).$$

We can see that for any $\tilde{D} \in \mathcal{D}$ we have

$$V_E(\tilde{\theta}) - V_E(\tilde{D}) = E_p(\tilde{\theta} - \tilde{D}) - \inf_{\pi \in \Pi_E} E_\pi(\tilde{\theta} - \tilde{D}) \geq 0$$

and therefore $V_E(\tilde{\theta}) = \sup_{\tilde{D} \in \mathcal{G}} V_E(\tilde{D})$ and it optimal to sell the whole firm. ■

Proof of Proposition 6

Let us start with the case $\theta_d \leq I \leq E_{p-\varepsilon_F}(\tilde{\theta})$. Notice first that in this case, the set \mathcal{H} is above the 45 degree line and thus any $\tilde{D} \in \mathcal{H}$ satisfies $D_d \leq D_u$. Thus each time E 's finances the project with $\tilde{D} \in \mathcal{H}$, he will get the proceeds $E_{p-\varepsilon_F}(\tilde{D})$. Now, it is convenient to decompose \mathcal{H} as

$$\mathcal{H} = \bigcup_{I \leq \gamma \leq E_{p-\varepsilon_F}(\tilde{\theta})} \mathcal{H}_\gamma, \text{ where } \mathcal{H}_\gamma = \left\{ \tilde{D} \mid 0 \leq \tilde{D} \leq \tilde{\theta} \text{ and } E_{p-\varepsilon_F}(\tilde{D}) = \gamma \right\}.$$

Let us first prove that any $\tilde{D} \in \mathcal{H}_\gamma$ is dominated by $\tilde{D}^\#$ which is the unique element of \mathcal{H}_γ satisfying $D_d^\# = \theta_d$. Using the fact that both \tilde{D} and $\tilde{D}^\#$ belong to \mathcal{H}_γ , straightforward calculations show that $\tilde{D}^\# = (D_u - \alpha, \theta_d)$ where $\alpha > 0$ solves the equation

$$\alpha(p - \varepsilon_F) - (\theta_d - D_d)(1 - p + \varepsilon_F) = 0.$$

For any prior $\pi \in \Pi_E$,

$$V_E^\pi(\tilde{D}^\#) - V_E^\pi(\tilde{D}) = E_\pi(\tilde{D}) - E_\pi(\tilde{D}^\#) = \pi\alpha - (1 - \pi)(\theta_d - D_d)$$

Because $\Pi_E \subseteq \Pi_F$, we have $\pi \geq p - \varepsilon_F$ for any $\pi \in \Pi_E$ and comparing the last two equalities we see that $V_E^\pi(\tilde{D}^\#) - V_E^\pi(\tilde{D}) \geq 0$. We conclude that $\tilde{D}^\#$ is preferred to \tilde{D} . The second step is to show that any $\tilde{D} \in \mathcal{H}_\gamma \cap \{\tilde{D} \mid D_d = \theta_d\}$ for some $\gamma \in [I, E_{p-\varepsilon_F}(\tilde{\theta})]$ is dominated by \tilde{D}^* . First observe that financing the firm with a security $\tilde{D} \in \mathcal{H}_\gamma \cap \{\tilde{D} \mid D_d = \theta_d\}$ yield the utility

$$V_E^\pi(\tilde{D}) = -I + E_{p-\varepsilon_F}(\tilde{D}) + E_\pi(\tilde{\theta} - \tilde{D}) = -I + \gamma + E_\pi(\tilde{\theta} - \tilde{D})$$

for any $\pi \in \Pi_E$. On the other hand if E finances the project with \tilde{D}^* , he gets the utility

$$V_E^\pi(\tilde{D}^*) = -I + E_{p-\varepsilon_F}(\tilde{D}^*) + E_\pi(\tilde{\theta} - \tilde{D}^*) = E_\pi(\tilde{\theta} - \tilde{D}^*)$$

for any any $\pi \in \Pi_E$. Therefore

$$V_E^\pi(\tilde{D}^*) - V_E^\pi(\tilde{D}) = I - \gamma + E_\pi(\tilde{D}^* - \tilde{D}) = I - \gamma + \pi(D_u - D_u^*)$$

Using the fact that $(\tilde{D}^*, \tilde{D}) \in \mathcal{H}_I \times \mathcal{H}_\gamma$ and the fact that $D_d^* = D_d = \theta_d$ we get

$$V_E^\pi(\tilde{D}^*) - V_E^\pi(\tilde{D}) = (\gamma - I) \left[\frac{\pi}{p - \varepsilon_F} - 1 \right]$$

which is positive for any $\pi \in \Pi_E$, again because $\pi \geq p - \varepsilon_F$ (recall that $\Pi_E \subseteq \Pi_F$). To summarize, using the transitivity of CEU, we have shown that \tilde{D}^* is preferred to any other implementable contract in (\mathcal{H}) and thus we can consider that the only “stable” contract is \tilde{D}^* .

Let us now turn to the case $I \leq \theta_d$. This case is different because the 45 degrees crosses the set \mathcal{H} and splits it into two subsets

$$\mathcal{H} = \mathcal{H}^+ \cup \mathcal{H}^-$$

where \mathcal{H}^+ (resp. \mathcal{H}^-) contains all the elements $\tilde{D} \in \mathcal{H}$ satisfying $D_u \geq D_d$ (resp. $D_u < D_d$). We will show here that while E is happy to finance the project with any security in the set \mathcal{H} he still prefers to finance the firm with a risk free security.

First, let us mention that if E finances the project with a security of the form $\tilde{D} = (\gamma, \gamma)$ with $\gamma \in [I, \theta_d]$ then he will get the utility

$$V_E^\pi(\tilde{D}) = -I + E_\pi(\tilde{\theta})$$

under the prior π . As a result, E is indifferent (prior by prior) between any two risk free contracts in \mathcal{H} . We will now focus on showing the dominance of the contract $\tilde{D}^f = (\theta_d, \theta_d)$ over all other contracts.

If E finances the project with contract $\tilde{D} \in \mathcal{H}^+$, the financiers will pay $\inf_{\pi \in \Pi_F} E_\pi(\tilde{D}) = E_{p-\varepsilon_F}(\tilde{D})$ and E gets the utility

$$V_E^\pi(\tilde{D}) = -I + E_{p-\varepsilon_F}(\tilde{D}) + E_\pi(\tilde{\theta} - \tilde{D})$$

for any prior $\pi \in \Pi_E$.

On the other hand, if E instead finances the project with the risk free security \tilde{D}^f he will get the utility

$$V_E^\pi(\tilde{D}^f) = -I + E_\pi(\tilde{\theta})$$

for any prior $\pi \in \Pi_E$ and thus

$$V_E^\pi(\tilde{D}^f) - V_E^\pi(\tilde{D}) = E_\pi(\tilde{D}) - E_{p-\varepsilon_F}(\tilde{D}).$$

Noticing that $\pi \geq p - \varepsilon_F$ and $D_u \geq D_d$ yields

$$V_E^\pi(\tilde{D}^f) \geq V_E^\pi(\tilde{D}) \text{ for all } \pi \in \Pi_E$$

meaning that E prefers \tilde{D}^f to any contract in \mathcal{H}^+ .

Now, if E finances the project with $\tilde{D} \in \mathcal{H}^-$, the financier F pays $\inf_{\pi \in \Pi_F} E_\pi(\tilde{D}) = E_{p+\varepsilon_F}(\tilde{D})$ he gets the utility

$$V_E^\pi(\tilde{D}) = -I + E_{p+\varepsilon_F}(\tilde{D}) + E_\pi(\tilde{\theta} - \tilde{D})$$

for any prior $\pi \in \Pi_E$. Thus

$$V_E^\pi(\tilde{D}^f) - V_E^\pi(\tilde{D}) = E_\pi(\tilde{D}) - E_{p+\varepsilon_F}(\tilde{D})$$

and recalling that $\pi \leq p + \varepsilon$ and $D_u \leq D_d$ gives

$$V_E^\pi(\tilde{D}^f) \geq V_E^\pi(\tilde{D}) \text{ for all } \pi \in \Pi_E.$$

■

Proof of Proposition 7

To be added.

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